11.8 Taylor and Maclaurin Series

This section shows how functions that are infinitely differentiable generate power series called Taylor series. In many cases, these series can provide useful polynomial approximations of the generating functions.

Series Representations

We know from Theorem 19 that within its interval of convergence the sum of a power series is a continuous function with derivatives of all orders. But what about the other way around? If a function $f(x)$ has derivatives of all orders on an interval I, can it be expressed as a power series on I? And if it can, what will its coefficients be?

We can answer the last question readily if we assume that $f(x)$ is the sum of a power series

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n(x-a)^n \\
&= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots
\end{aligned}
$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I we obtain

$$
\begin{aligned}
f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots \\
f''(x) &= 1\cdot 2a_2 + 2\cdot 3a_3(x-a) + 3\cdot 4a_4(x-a)^2 + \cdots \\
f'''(x) &= 1\cdot 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4(x-a) + 3\cdot 4\cdot 5a_5(x-a)^2 + \cdots,
\end{aligned}
$$

with the nth derivative, for all n, being

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x - a) \text{ as a factor.}$$

Since these equations all hold at $x = a$, we have

$$\begin{aligned} f'(a) &= a_1, \\ f''(a) &= 1 \cdot 2a_2, \\ f'''(a) &= 1 \cdot 2 \cdot 3a_3, \end{aligned}$$

and, in general,

$$f^{(n)}(a) = n!a_n.$$

These formulas reveal a pattern in the coefficients of any power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ that converges to the values of f on I ("represents f on I"). If there *is* such a series (still an open question), then there is only one such series and its nth coefficient is

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

If f has a series representation, then the series must be

$$\begin{aligned} f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 \\ + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots. \qquad (1) \end{aligned}$$

But if we start with an arbitrary function f that is infinitely differentiable on an interval I centered at $x = a$ and use it to generate the series in Equation (1), will the series then converge to $f(x)$ at each x in the interior of I? The answer is maybe—for some functions it will but for other functions it will not, as we will see.

Taylor and Maclaurin Series

HISTORICAL BIOGRAPHIES

Brook Taylor
(1685–1731)

Colin Maclaurin
(1698–1746)

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 \\ + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots. \end{aligned}$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

The Maclaurin series generated by f is often just called the Taylor series of f.

EXAMPLE 1 Finding a Taylor Series

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$
\begin{aligned}
f(x) &= x^{-1}, & f(2) &= 2^{-1} = \frac{1}{2}, \\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2}, \\
f''(x) &= 2!x^{-3}, & \frac{f''(2)}{2!} &= 2^{-3} = \frac{1}{2^3}, \\
f'''(x) &= -3!x^{-4}, & \frac{f'''(2)}{3!} &= -\frac{1}{2^4}, \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n n! x^{-(n+1)}, & \frac{f^{(n)}(2)}{n!} &= \frac{(-1)^n}{2^{n+1}}.
\end{aligned}
$$

The Taylor series is

$$
\begin{aligned}
f(2) &+ f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\
&= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.
\end{aligned}
$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$. ■

Taylor Polynomials

The linearization of a differentiable function f at a point a is the polynomial of degree one given by

$$P_1(x) = f(a) + f'(a)(x-a).$$

In Section 3.8 we used this linearization to approximate $f(x)$ at values of x near a. If f has derivatives of higher order at a, then it has higher-order polynomial approximations as well, one for each available derivative. These polynomials are called the Taylor polynomials of f.

DEFINITION Taylor Polynomial of Order n

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We speak of a Taylor polynomial of *order* n rather than *degree* n because $f^{(n)}(a)$ may be zero. The first two Taylor polynomials of $f(x) = \cos x$ at $x = 0$, for example, are $P_0(x) = 1$ and $P_1(x) = 1$. The first-order Taylor polynomial has degree zero, not one.

Just as the linearization of f at $x = a$ provides the best linear approximation of f in the neighborhood of a, the higher-order Taylor polynomials provide the best polynomial approximations of their respective degrees. (See Exercise 32.)

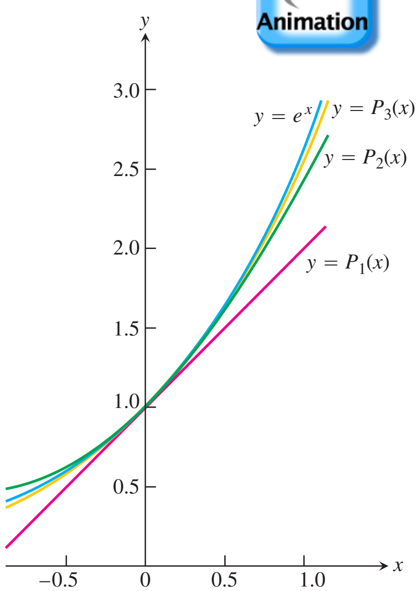

FIGURE 11.12 The graph of $f(x) = e^x$ and its Taylor polynomials
$P_1(x) = 1 + x$
$P_2(x) = 1 + x + (x^2/2!)$
$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.
Notice the very close agreement near the center $x = 0$ (Example 2).

EXAMPLE 2 Finding Taylor Polynomials for e^x

Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution Since

$$f(x) = e^x, \quad f'(x) = e^x, \quad \ldots, \quad f^{(n)}(x) = e^x, \quad \ldots,$$

we have

$$f(0) = e^0 = 1, \quad f'(0) = 1, \quad \ldots, \quad f^{(n)}(0) = 1, \quad \ldots.$$

The Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This is also the Maclaurin series for e^x. In Section 11.9 we will see that the series converges to e^x at every x.

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

See Figure 11.12. ■

EXAMPLE 3 Finding Taylor Polynomials for $\cos x$

Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$
\begin{aligned}
f(x) &= \cos x, & f'(x) &= -\sin x,\\
f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x,\\
&\vdots & &\vdots\\
f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x.
\end{aligned}
$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$
\begin{aligned}
f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
= 1 + 0\cdot x - \frac{x^2}{2!} + 0\cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots \\
= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}.
\end{aligned}
$$

This is also the Maclaurin series for $\cos x$. In Section 11.9, we will see that the series converges to $\cos x$ at every x.

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n + 1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!}.$$

Figure 11.13 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y-axis. ■

Animation

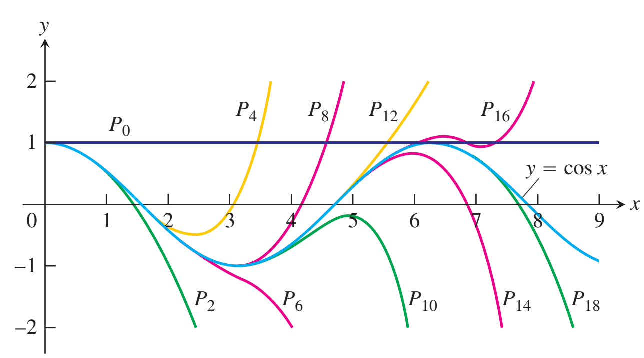

FIGURE 11.13 The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n}\frac{(-1)^k x^{2k}}{(2k)!}$$

converge to $\cos x$ as $n \to \infty$. We can deduce the behavior of $\cos x$ arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

EXAMPLE 4 A Function f Whose Taylor Series Converges at Every x but Converges to $f(x)$ Only at $x = 0$

It can be shown (though not easily) that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

(Figure 11.14) has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n. This means that the Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$
$$= 0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots$$
$$= 0 + 0 + \cdots + 0 + \cdots.$$

The series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$. ■

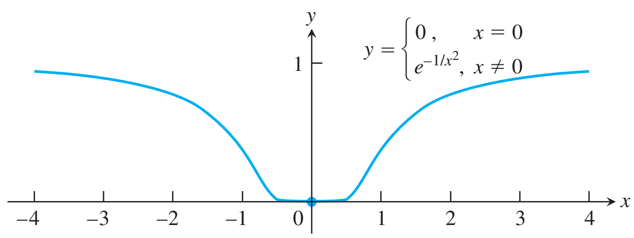

FIGURE 11.14 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4).

Two questions still remain.

1. For what values of x can we normally expect a Taylor series to converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

The answers are provided by a theorem of Taylor in the next section.