EXERCISES 11.10

Binomial Series

Find the first four terms of the binomial series for the functions in Exercises 1–10.

1. $(1 + x)^{1/2}$

2. $(1 + x)^{1/3}$

3. $(1 - x)^{-1/2}$

4. $(1 - 2x)^{1/2}$

5. $\left(1 + \frac{x}{2}\right)^{-2}$

6. $\left(1 - \frac{x}{2}\right)^{-2}$

7. $(1 + x^3)^{-1/2}$

8. $(1 + x^2)^{-1/3}$

9. $\left(1 + \frac{1}{x}\right)^{1/2}$

10. $\left(1 - \frac{2}{x}\right)^{1/3}$

Find the binomial series for the functions in Exercises 11–14.

11. $(1 + x)^4$

12. $(1 + x^2)^3$

13. $(1-2x)^3$ **14.** $\left(1-\frac{x}{2}\right)^4$

Initial Value Problems

Find series solutions for the initial value problems in Exercises 15–32.

15. $y' + y = 0, \quad y(0) = 1$ **16.** $y' - 2y = 0, \quad y(0) = 1$

17. $y' - y = 1, \quad y(0) = 0$ **18.** $y' + y = 1, \quad y(0) = 2$

19. $y' - y = x, \quad y(0) = 0$ **20.** $y' + y = 2x, \quad y(0) = -1$

21. $y' - xy = 0, \quad y(0) = 1$ **22.** $y' - x^2y = 0, \quad y(0) = 1$

23. $(1-x)y' - y = 0, \quad y(0) = 2$

24. $(1+x^2)y' + 2xy = 0, \quad y(0) = 3$

25. $y'' - y = 0, \quad y'(0) = 1$ and $y(0) = 0$

26. $y'' + y = 0, \quad y'(0) = 0$ and $y(0) = 1$

27. $y'' + y = x, \quad y'(0) = 1$ and $y(0) = 2$

28. $y'' - y = x, \quad y'(0) = 2$ and $y(0) = -1$

29. $y'' - y = -x, \quad y'(2) = -2$ and $y(2) = 0$

30. $y'' - x^2y = 0, \quad y'(0) = b$ and $y(0) = a$

31. $y'' + x^2y = x, \quad y'(0) = b$ and $y(0) = a$

32. $y'' - 2y' + y = 0, \quad y'(0) = 1$ and $y(0) = 0$

Approximations and Nonelementary Integrals

T In Exercises 33–36, use series to estimate the integrals' values with an error of magnitude less than 10^{-3}. (The answer section gives the integrals' values rounded to five decimal places.)

33. $\int_0^{0.2} \sin x^2\, dx$ **34.** $\int_0^{0.2} \frac{e^{-x}-1}{x}\, dx$

35. $\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\, dx$ **36.** $\int_0^{0.25} \sqrt[3]{1+x^2}\, dx$

T Use series to approximate the values of the integrals in Exercises 37–40 with an error of magnitude less than 10^{-8}.

37. $\int_0^{0.1} \frac{\sin x}{x}\, dx$ **38.** $\int_0^{0.1} e^{-x^2}\, dx$

39. $\int_0^{0.1} \sqrt{1+x^4}\, dx$ **40.** $\int_0^{1} \frac{1-\cos x}{x^2}\, dx$

41. Estimate the error if $\cos t^2$ is approximated by $1 - \frac{t^4}{2} + \frac{t^8}{4!}$ in the integral $\int_0^1 \cos t^2\, dt$.

42. Estimate the error if $\cos\sqrt{t}$ is approximated by $1 - \frac{t}{2} + \frac{t^2}{4!} - \frac{t^3}{6!}$ in the integral $\int_0^1 \cos\sqrt{t}\, dt$.

In Exercises 43–46, find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3}.

43. $F(x) = \int_0^x \sin t^2\, dt, \quad [0, 1]$

44. $F(x) = \int_0^x t^2 e^{-t^2}\, dt, \quad [0, 1]$

45. $F(x) = \int_0^x \tan^{-1} t\, dt, \quad$ **(a)** $[0, 0.5]$ **(b)** $[0, 1]$

46. $F(x) = \int_0^x \frac{\ln(1+t)}{t}\, dt, \quad$ **(a)** $[0, 0.5]$ **(b)** $[0, 1]$

Indeterminate Forms

Use series to evaluate the limits in Exercises 47–56.

47. $\lim_{x\to 0} \frac{e^x - (1+x)}{x^2}$ **48.** $\lim_{x\to 0} \frac{e^x - e^{-x}}{x}$

49. $\lim_{t\to 0} \frac{1 - \cos t - (t^2/2)}{t^4}$ **50.** $\lim_{\theta\to 0} \frac{\sin\theta - \theta + (\theta^3/6)}{\theta^5}$

51. $\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3}$ **52.** $\lim_{y\to 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y}$

53. $\lim_{x\to\infty} x^2(e^{-1/x^2} - 1)$ **54.** $\lim_{x\to\infty} (x+1)\sin\frac{1}{x+1}$

55. $\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x}$ **56.** $\lim_{x\to 2} \frac{x^2-4}{\ln(x-1)}$

Theory and Examples

57. Replace x by $-x$ in the Taylor series for $\ln(1+x)$ to obtain a series for $\ln(1-x)$. Then subtract this from the Taylor series for $\ln(1+x)$ to show that for $|x| < 1$,

$$\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

58. How many terms of the Taylor series for $\ln(1+x)$ should you add to be sure of calculating $\ln(1.1)$ with an error of magnitude less than 10^{-8}? Give reasons for your answer.

59. According to the Alternating Series Estimation Theorem, how many terms of the Taylor series for $\tan^{-1} 1$ would you have to add to be sure of finding $\pi/4$ with an error of magnitude less than 10^{-3}? Give reasons for your answer.

60. Show that the Taylor series for $f(x) = \tan^{-1} x$ diverges for $|x| > 1$.

T **61. Estimating Pi** About how many terms of the Taylor series for $\tan^{-1} x$ would you have to use to evaluate each term on the right-hand side of the equation

$$\pi = 48\tan^{-1}\frac{1}{18} + 32\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239}$$

with an error of magnitude less than 10^{-6}? In contrast, the convergence of $\sum_{n=1}^{\infty}(1/n^2)$ to $\pi^2/6$ is so slow that even 50 terms will not yield two-place accuracy.

62. Integrate the first three nonzero terms of the Taylor series for $\tan t$ from 0 to x to obtain the first three nonzero terms of the Taylor series for $\ln \sec x$.

63. a. Use the binomial series and the fact that

$$\frac{d}{dx}\sin^{-1}x = (1 - x^2)^{-1/2}$$

to generate the first four nonzero terms of the Taylor series for $\sin^{-1}x$. What is the radius of convergence?

b. Series for $\cos^{-1}x$ Use your result in part (a) to find the first five nonzero terms of the Taylor series for $\cos^{-1}x$.

64. a. Series for $\sinh^{-1}x$ Find the first four nonzero terms of the Taylor series for

$$\sinh^{-1}x = \int_0^x \frac{dt}{\sqrt{1 + t^2}}.$$

T **b.** Use the first *three* terms of the series in part (a) to estimate $\sinh^{-1}0.25$. Give an upper bound for the magnitude of the estimation error.

65. Obtain the Taylor series for $1/(1 + x)^2$ from the series for $-1/(1 + x)$.

66. Use the Taylor series for $1/(1 - x^2)$ to obtain a series for $2x/(1 - x^2)^2$.

T **67. Estimating Pi** The English mathematician Wallis discovered the formula

$$\frac{\pi}{4} = \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot \cdots}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot \cdots}.$$

Find π to two decimal places with this formula.

T **68.** Construct a table of natural logarithms $\ln n$ for $n = 1, 2, 3, \ldots, 10$ by using the formula in Exercise 57, but taking advantage of the relationships $\ln 4 = 2\ln 2$, $\ln 6 = \ln 2 + \ln 3$, $\ln 8 = 3\ln 2$, $\ln 9 = 2\ln 3$, and $\ln 10 = \ln 2 + \ln 5$ to reduce the job to the calculation of relatively few logarithms by series. Start by using the following values for x in Exercise 57:

$$\frac{1}{3}, \quad \frac{1}{5}, \quad \frac{1}{9}, \quad \frac{1}{13}.$$

69. Series for $\sin^{-1}x$ Integrate the binomial series for $(1 - x^2)^{-1/2}$ to show that for $|x| < 1$,

$$\sin^{-1}x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \frac{x^{2n+1}}{2n+1}.$$

70. Series for $\tan^{-1}x$ for $|x| > 1$ Derive the series

$$\tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad x > 1$$

$$\tan^{-1}x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad x < -1,$$

by integrating the series

$$\frac{1}{1+t^2} = \frac{1}{t^2} \cdot \frac{1}{1 + (1/t^2)} = \frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \cdots$$

in the first case from x to ∞ and in the second case from $-\infty$ to x.

71. The value of $\sum_{n=1}^{\infty}\tan^{-1}(2/n^2)$

a. Use the formula for the tangent of the difference of two angles to show that

$$\tan\left(\tan^{-1}(n+1) - \tan^{-1}(n-1)\right) = \frac{2}{n^2}$$

b. Show that

$$\sum_{n=1}^{N} \tan^{-1}\frac{2}{n^2} = \tan^{-1}(N+1) + \tan^{-1}N - \frac{\pi}{4}.$$

c. Find the value of $\sum_{n=1}^{\infty}\tan^{-1}\frac{2}{n^2}$.