11.11 Fourier Series

HISTORICAL BIOGRAPHY

Jean-Baptiste Joseph Fourier

(1766–1830)

We have seen how Taylor series can be used to approximate a function f by polynomials. The Taylor polynomials give a close fit to f near a particular point $x = a$, but the error in the approximation can be large at points that are far away. There is another method that often gives good approximations on wide intervals, and often works with discontinuous functions for which Taylor polynomials fail. Introduced by Joseph Fourier, this method approximates functions with sums of sine and cosine functions. It is well suited for analyzing periodic functions, such as radio signals and alternating currents, for solving heat transfer problems, and for many other problems in science and engineering.

Suppose we wish to approximate a function f on the interval $[0, 2\pi]$ by a sum of sine and cosine functions,

$$f_n(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx)$$

or, in sigma notation,

$$f_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \tag{1}$$

We would like to choose values for the constants $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ that make $f_n(x)$ a "best possible" approximation to $f(x)$. The notion of "best possible" is defined as follows:

1. $f_n(x)$ and $f(x)$ give the same value when integrated from 0 to 2π.
2. $f_n(x) \cos kx$ and $f(x) \cos kx$ give the same value when integrated from 0 to 2π $(k = 1, \ldots, n)$.
3. $f_n(x) \sin kx$ and $f(x) \sin kx$ give the same value when integrated from 0 to 2π $(k = 1, \ldots, n)$.

Altogether we impose $2n + 1$ conditions on f_n:

$$\int_0^{2\pi} f_n(x)\,dx = \int_0^{2\pi} f(x)\,dx,$$

$$\int_0^{2\pi} f_n(x) \cos kx\,dx = \int_0^{2\pi} f(x) \cos kx\,dx, \qquad k = 1, \ldots, n,$$

$$\int_0^{2\pi} f_n(x) \sin kx\,dx = \int_0^{2\pi} f(x) \sin kx\,dx, \qquad k = 1, \ldots, n.$$

It is possible to choose $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ so that all these conditions are satisfied, by proceeding as follows. Integrating both sides of Equation (1) from 0 to 2π gives

$$\int_0^{2\pi} f_n(x)\,dx = 2\pi a_0$$

since the integral over $[0, 2\pi]$ of $\cos kx$ equals zero when $k \geq 1$, as does the integral of $\sin kx$. Only the constant term a_0 contributes to the integral of f_n over $[0, 2\pi]$. A similar calculation applies with each of the other terms. If we multiply both sides of Equation (1) by $\cos x$ and integrate from 0 to 2π then we obtain

$$\int_0^{2\pi} f_n(x) \cos x\,dx = \pi a_1.$$

This follows from the fact that

$$\int_0^{2\pi} \cos px \cos px\,dx = \pi$$

and

$$\int_0^{2\pi} \cos px \cos qx\,dx = \int_0^{2\pi} \cos px \sin mx\,dx = \int_0^{2\pi} \sin px \sin qx\,dx = 0$$

whenever p, q and m are integers and p is not equal to q (Exercises 9–13). If we multiply Equation (1) by $\sin x$ and integrate from 0 to 2π we obtain

$$\int_0^{2\pi} f_n(x)\sin x\,dx = \pi b_1.$$

Proceeding in a similar fashion with

$$\cos 2x, \sin 2x, \ldots, \cos nx, \sin nx$$

we obtain only one nonzero term each time, the term with a sine-squared or cosine-squared term. To summarize,

$$\int_0^{2\pi} f_n(x)\,dx = 2\pi a_0$$

$$\int_0^{2\pi} f_n(x)\cos kx\,dx = \pi a_k, \qquad k = 1, \ldots, n$$

$$\int_0^{2\pi} f_n(x)\sin kx\,dx = \pi b_k, \qquad k = 1, \ldots, n$$

We chose f_n so that the integrals on the left remain the same when f_n is replaced by f, so we can use these equations to find $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ from f:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \tag{2}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad k = 1, \ldots, n \tag{3}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx, \qquad k = 1, \ldots, n \tag{4}$$

The only condition needed to find these coefficients is that the integrals above must exist. If we let $n \to \infty$ and use these rules to get the coefficients of an infinite series, then the resulting sum is called the **Fourier series for $f(x)$**,

$$a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx). \tag{5}$$

EXAMPLE 1 Finding a Fourier Series Expansion

Fourier series can be used to represent some functions that cannot be represented by Taylor series; for example, the step function f shown in Figure 11.16a.

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ 2, & \text{if } \pi < x \le 2\pi. \end{cases}$$

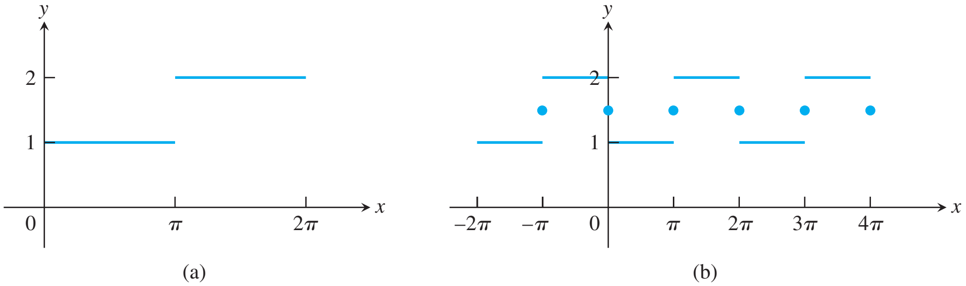

FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$$

(b) The graph of the Fourier series for f is periodic and has the value $3/2$ at each point of discontinuity (Example 1).

The coefficients of the Fourier series of f are computed using Equations (2), (3), and (4).

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\left(\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} 2\,dx\right) = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \\ &= \frac{1}{\pi}\left(\int_0^{\pi} \cos kx\,dx + \int_{\pi}^{2\pi} 2\cos kx\,dx\right) \\ &= \frac{1}{\pi}\left(\left[\frac{\sin kx}{k}\right]_0^{\pi} + \left[\frac{2\sin kx}{k}\right]_{\pi}^{2\pi}\right) = 0, \qquad k \ge 1 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx \\ &= \frac{1}{\pi}\left(\int_0^{\pi} \sin kx\,dx + \int_{\pi}^{2\pi} 2\sin kx\,dx\right) \\ &= \frac{1}{\pi}\left(\left[-\frac{\cos kx}{k}\right]_0^{\pi} + \left[-\frac{2\cos kx}{k}\right]_{\pi}^{2\pi}\right) \\ &= \frac{\cos k\pi - 1}{k\pi} = \frac{(-1)^k - 1}{k\pi}. \end{aligned}$$

So

$$a_0 = \frac{3}{2}, \quad a_1 = a_2 = \cdots = 0,$$

and

$$b_1 = -\frac{2}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{2}{3\pi}, \quad b_4 = 0, \quad b_5 = -\frac{2}{5\pi}, \quad b_6 = 0, \ldots$$

The Fourier series is

$$\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Notice that at $x = \pi$, where the function $f(x)$ jumps from 1 to 2, all the sine terms vanish, leaving $3/2$ as the value of the series. This is not the value of f at π, since $f(\pi) = 1$. The Fourier series also sums to $3/2$ at $x = 0$ and $x = 2\pi$. In fact, all terms in the Fourier series are periodic, of period 2π, and the value of the series at $x + 2\pi$ is the same as its value at x. The series we obtained represents the periodic function graphed in Figure 11.16b, with domain the entire real line and a pattern that repeats over every interval of width 2π. The function jumps discontinuously at $x = n\pi, n = 0, \pm 1, \pm 2, \ldots$ and at these points has value $3/2$, the average value of the one-sided limits from each side. The convergence of the Fourier series of f is indicated in Figure 11.17. ■

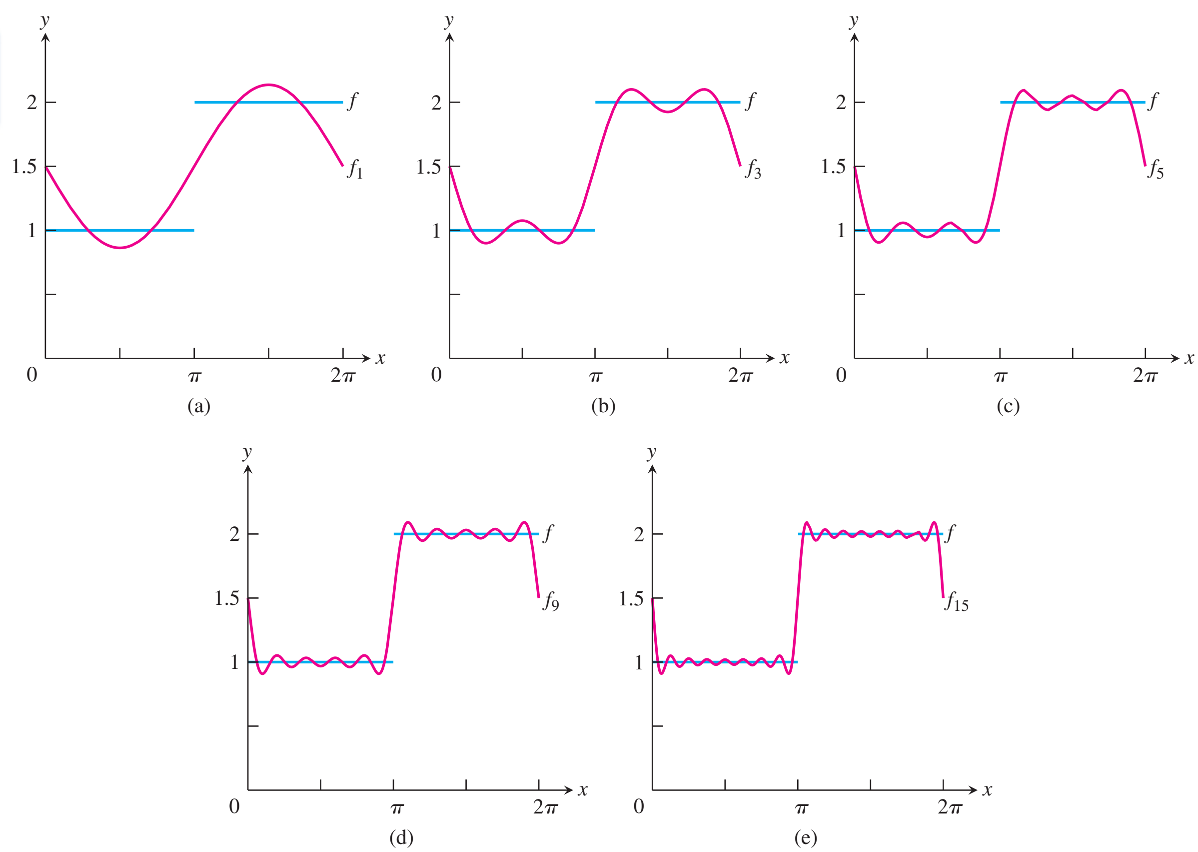

FIGURE 11.17 The Fourier approximation functions f_1, f_3, f_5, f_9, and f_{15} of the function $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$ in Example 1.

Convergence of Fourier Series

Taylor series are computed from the value of a function and its derivatives at a single point $x = a$, and cannot reflect the behavior of a discontinuous function such as f in Example 1 past a discontinuity. The reason that a Fourier series can be used to represent such functions is that the Fourier series of a function depends on the existence of certain *integrals*, whereas the Taylor series depends on derivatives of a function near a single point. A function can be fairly "rough," even discontinuous, and still be integrable.

The coefficients used to construct Fourier series are precisely those one should choose to minimize the integral of the square of the error in approximating f by f_n. That is,

$$\int_0^{2\pi} [f(x) - f_n(x)]^2 \, dx$$

is minimized by choosing $a_0, a_1, a_2, \dots a_n$ and $b_1, b_2, \dots, b_n$ as we did. While Taylor series are useful to approximate a function and its derivatives near a point, Fourier series minimize an error which is distributed over an interval.

We state without proof a result concerning the convergence of Fourier series. A function is **piecewise continuous** over an interval I if it has finitely many discontinuities on the interval, and at these discontinuities one-sided limits exist from each side. (See Chapter 5, Additional Exercises 11–18.)

THEOREM 24 Let $f(x)$ be a function such that f and f' are piecewise continuous on the interval $[0, 2\pi]$. Then f is equal to its Fourier series at all points where f is continuous. At a point c where f has a discontinuity, the Fourier series converges to

$$\frac{f(c^+) + f(c^-)}{2}$$

where $f(c^+)$ and $f(c^-)$ are the right- and left-hand limits of f at c.