Chapter 11 Practice Exercises

Convergent or Divergent Sequences

Which of the sequences whose nth terms appear in Exercises 1–18 converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = 1 + \frac{(-1)^n}{n}$

2. $a_n = \frac{1 - (-1)^n}{\sqrt{n}}$

3. $a_n = \frac{1 - 2^n}{2^n}$

4. $a_n = 1 + (0.9)^n$

5. $a_n = \sin \frac{n\pi}{2}$

6. $a_n = \sin n\pi$

7. $a_n = \frac{\ln (n^2)}{n}$

8. $a_n = \frac{\ln (2n + 1)}{n}$

9. $a_n = \frac{n + \ln n}{n}$

10. $a_n = \frac{\ln (2n^3 + 1)}{n}$

11. $a_n = \left(\frac{n - 5}{n}\right)^n$

12. $a_n = \left(1 + \frac{1}{n}\right)^{-n}$

13. $a_n = \sqrt[n]{\frac{3^n}{n}}$

14. $a_n = \left(\frac{3}{n}\right)^{1/n}$

15. $a_n = n(2^{1/n} - 1)$

16. $a_n = \sqrt[n]{2n + 1}$

17. $a_n = \frac{(n + 1)!}{n!}$

18. $a_n = \frac{(-4)^n}{n!}$

Convergent Series

Find the sums of the series in Exercises 19–24.

19. $\sum_{n=3}^{\infty} \frac{1}{(2n - 3)(2n - 1)}$

20. $\sum_{n=2}^{\infty} \frac{-2}{n(n + 1)}$

21. $\sum_{n=1}^{\infty} \frac{9}{(3n - 1)(3n + 2)}$

22. $\sum_{n=3}^{\infty} \frac{-8}{(4n - 3)(4n + 1)}$

23. $\sum_{n=0}^{\infty} e^{-n}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n}$

Convergent or Divergent Series

Which of the series in Exercises 25–40 converge absolutely, which converge conditionally, and which diverge? Give reasons for your answers.

25. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

26. $\sum_{n=1}^{\infty} \frac{-5}{n}$

27. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

28. $\sum_{n=1}^{\infty} \frac{1}{2n^3}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln (n + 1)}$

30. $\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$

31. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

32. $\sum_{n=3}^{\infty} \frac{\ln n}{\ln (\ln n)}$

33. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2 + 1}}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3 + 1}$

35. $\sum_{n=1}^{\infty} \frac{n + 1}{n!}$

36. $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2 + 1)}{2n^2 + n - 1}$

37. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

38. $\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$

39. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n + 1)(n + 2)}}$

40. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$

Power Series

In Exercises 41–50, **(a)** find the series' radius and interval of convergence. Then identify the values of x for which the series converges **(b)** absolutely and **(c)** conditionally.

41. $\sum_{n=1}^{\infty} \frac{(x + 4)^n}{n3^n}$

42. $\sum_{n=1}^{\infty} \frac{(x - 1)^{2n-2}}{(2n - 1)!}$

43. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x - 1)^n}{n^2}$

44. $\sum_{n=0}^{\infty} \frac{(n + 1)(2x + 1)^n}{(2n + 1)2^n}$

45. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

46. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

47. $\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$

48. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{2n+1}}{2n+1}$

49. $\sum_{n=1}^{\infty} (\operatorname{csch} n)x^n$

50. $\sum_{n=1}^{\infty} (\coth n)x^n$

Maclaurin Series

Each of the series in Exercises 51–56 is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

51. $1 - \frac{1}{4} + \frac{1}{16} - \cdots + (-1)^n \frac{1}{4^n} + \cdots$

52. $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \cdots + (-1)^{n-1} \frac{2^n}{n3^n} + \cdots$

53. $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \cdots$

54. $1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} - \cdots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \cdots$

55. $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$

56. $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \cdots$

$$+ (-1)^{n-1} \frac{1}{(2n-1)\left(\sqrt{3}\right)^{2n-1}} + \cdots$$

Find Taylor series at $x = 0$ for the functions in Exercises 57–64.

57. $\frac{1}{1-2x}$

58. $\frac{1}{1+x^3}$

59. $\sin \pi x$

60. $\sin \frac{2x}{3}$

61. $\cos (x^{5/2})$

62. $\cos \sqrt{5x}$

63. $e^{(\pi x/2)}$

64. e^{-x^2}

Taylor Series

In Exercises 65–68, find the first four nonzero terms of the Taylor series generated by f at $x = a$.

65. $f(x) = \sqrt{3 + x^2}$ at $x = -1$

66. $f(x) = 1/(1 - x)$ at $x = 2$

67. $f(x) = 1/(x + 1)$ at $x = 3$

68. $f(x) = 1/x$ at $x = a > 0$

Initial Value Problems

Use power series to solve the initial value problems in Exercises 69–76.

69. $y' + y = 0, \quad y(0) = -1$

70. $y' - y = 0, \quad y(0) = -3$

71. $y' + 2y = 0, \quad y(0) = 3$

72. $y' + y = 1, \quad y(0) = 0$

73. $y' - y = 3x, \quad y(0) = -1$

74. $y' + y = x, \quad y(0) = 0$

75. $y' - y = x, \quad y(0) = 1$

76. $y' - y = -x, \quad y(0) = 2$

Nonelementary Integrals

Use series to approximate the values of the integrals in Exercises 77–80 with an error of magnitude less than 10^{-8}. (The answer section gives the integrals' values rounded to 10 decimal places.)

77. $\int_0^{1/2} e^{-x^3}\, dx$

78. $\int_0^1 x \sin (x^3)\, dx$

79. $\int_0^{1/2} \frac{\tan^{-1} x}{x}\, dx$

80. $\int_0^{1/64} \frac{\tan^{-1} x}{\sqrt{x}}\, dx$

Indeterminate Forms

In Exercises 81–86:

a. Use power series to evaluate the limit.

T **b.** Then use a grapher to support your calculation.

81. $\lim_{x \to 0} \frac{7 \sin x}{e^{2x} - 1}$

82. $\lim_{\theta \to 0} \frac{e^\theta - e^{-\theta} - 2\theta}{\theta - \sin \theta}$

83. $\lim_{t \to 0} \left(\frac{1}{2 - 2\cos t} - \frac{1}{t^2} \right)$

84. $\lim_{h \to 0} \frac{(\sin h)/h - \cos h}{h^2}$

85. $\lim_{z \to 0} \frac{1 - \cos^2 z}{\ln (1 - z) + \sin z}$

86. $\lim_{y \to 0} \frac{y^2}{\cos y - \cosh y}$

87. Use a series representation of $\sin 3x$ to find values of r and s for which

$$\lim_{x \to 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s \right) = 0.$$

88. **a.** Show that the approximation $\csc x \approx 1/x + x/6$ in Section 11.10, Example 9, leads to the approximation $\sin x \approx 6x/(6 + x^2)$.

T **b.** Compare the accuracies of the approximations $\sin x \approx x$ and $\sin x \approx 6x/(6 + x^2)$ by comparing the graphs of $f(x) = \sin x - x$ and $g(x) = \sin x - (6x/(6 + x^2))$. Describe what you find.

Theory and Examples

89. **a.** Show that the series

$$\sum_{n=1}^{\infty} \left(\sin \frac{1}{2n} - \sin \frac{1}{2n + 1} \right)$$

converges.

T **b.** Estimate the magnitude of the error involved in using the sum of the sines through $n = 20$ to approximate the sum of the series. Is the approximation too large, or too small? Give reasons for your answer.

90. **a.** Show that the series $\sum_{n=1}^{\infty} \left(\tan \frac{1}{2n} - \tan \frac{1}{2n+1} \right)$ converges.

T **b.** Estimate the magnitude of the error in using the sum of the tangents through $-\tan (1/41)$ to approximate the sum of the series. Is the approximation too large, or too small? Give reasons for your answer.

91. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} x^n.$$

92. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}{4 \cdot 9 \cdot 14 \cdot \cdots \cdot (5n-1)} (x-1)^n.$$

93. Find a closed-form formula for the nth partial sum of the series $\sum_{n=2}^{\infty} \ln(1-(1/n^2))$ and use it to determine the convergence or divergence of the series.

94. Evaluate $\sum_{k=2}^{\infty}(1/(k^2-1))$ by finding the limits as $n \to \infty$ of the series' nth partial sum.

95. a. Find the interval of convergence of the series

$$y = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)}{(3n)!}x^{3n} + \cdots.$$

b. Show that the function defined by the series satisfies a differential equation of the form

$$\frac{d^2y}{dx^2} = x^a y + b$$

and find the values of the constants a and b.

96. a. Find the Maclaurin series for the function $x^2/(1+x)$.

b. Does the series converge at $x = 1$? Explain.

97. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} a_n b_n$? Give reasons for your answer.

98. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} a_n b_n$? Give reasons for your answer.

99. Prove that the sequence $\{x_n\}$ and the series $\sum_{k=1}^{\infty}(x_{k+1}-x_k)$ both converge or both diverge.

100. Prove that $\sum_{n=1}^{\infty}(a_n/(1+a_n))$ converges if $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges.

101. (*Continuation of Section 4.7, Exercise 27.*) If you did Exercise 27 in Section 4.7, you saw that in practice Newton's method stopped too far from the root of $f(x) = (x-1)^{40}$ to give a useful estimate of its value, $x = 1$. Prove that nevertheless, for any starting value $x_0 \neq 1$, the sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$ of approximations generated by Newton's method really does converge to 1.

102. Suppose that $a_1, a_2, a_3, \ldots, a_n$ are positive numbers satisfying the following conditions:

i. $a_1 \geq a_2 \geq a_3 \geq \cdots$;

ii. the series $a_2 + a_4 + a_8 + a_{16} + \cdots$ diverges.

Show that the series

$$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \cdots$$

diverges.

103. Use the result in Exercise 102 to show that

$$1 + \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

104. Suppose you wish to obtain a quick estimate for the value of $\int_0^1 x^2 e^x \, dx$. There are several ways to do this.

a. Use the Trapezoidal Rule with $n = 2$ to estimate $\int_0^1 x^2 e^x \, dx$.

b. Write out the first three nonzero terms of the Taylor series at $x = 0$ for $x^2 e^x$ to obtain the fourth Taylor polynomial $P(x)$ for $x^2 e^x$. Use $\int_0^1 P(x)\, dx$ to obtain another estimate for $\int_0^1 x^2 e^x \, dx$.

c. The second derivative of $f(x) = x^2 e^x$ is positive for all $x > 0$. Explain why this enables you to conclude that the Trapezoidal Rule estimate obtained in part (a) is too large. (*Hint:* What does the second derivative tell you about the graph of a function? How does this relate to the trapezoidal approximation of the area under this graph?)

d. All the derivatives of $f(x) = x^2 e^x$ are positive for $x > 0$. Explain why this enables you to conclude that all Maclaurin polynomial approximations to $f(x)$ for x in [0, 1] will be too small. (*Hint:* $f(x) = P_n(x) + R_n(x)$.)

e. Use integration by parts to evaluate $\int_0^1 x^2 e^x \, dx$.

Fourier Series

Find the Fourier series for the functions in Exercises 105–108. Sketch each function.

105. $f(x) = \begin{cases} 0, & 0 \leq x \leq \pi \\ 1, & \pi < x \leq 2\pi \end{cases}$

106. $f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 1, & \pi < x \leq 2\pi \end{cases}$

107. $f(x) = \begin{cases} \pi - x, & 0 \leq x \leq \pi \\ x - 2\pi, & \pi < x \leq 2\pi \end{cases}$

108. $f(x) = |\sin x|, \quad 0 \leq x \leq 2\pi$