# 12.3 The Dot Product

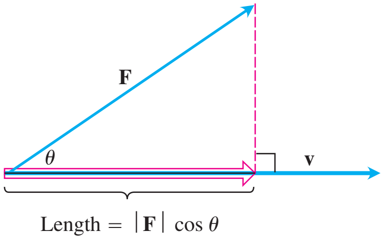



**FIGURE 12.18** The magnitude of the force **F** in the direction of vector **v** is the length $|\mathbf{F}|\cos\theta$ of the projection of **F** onto **v**.

If a force **F** is applied to a particle moving along a path, we often need to know the magnitude of the force in the direction of motion. If **v** is parallel to the tangent line to the path at the point where **F** is applied, then we want the magnitude of **F** in the direction of **v**. Figure 12.18 shows that the scalar quantity we seek is the length $|\mathbf{F}|\cos\theta$, where $\theta$ is the angle between the two vectors **F** and **v**.

In this section, we show how to calculate easily the angle between two vectors directly from their components. A key part of the calculation is an expression called the *dot product*. Dot products are also called *inner* or *scalar* products because the product results in a scalar, not a vector. After investigating the dot product, we apply it to finding the projection of one vector onto another (as displayed in Figure 12.18) and to finding the work done by a constant force acting through a displacement.

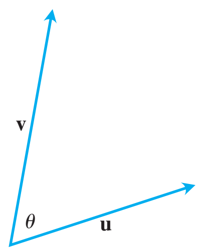



**FIGURE 12.19** The angle between **u** and **v**.

## Angle Between Vectors

When two nonzero vectors **u** and **v** are placed so their initial points coincide, they form an angle $\theta$ of measure $0 \leq \theta \leq \pi$ (Figure 12.19). If the vectors do not lie along the same line, the angle $\theta$ is measured in the plane containing both of them. If they do lie along the same line, the angle between them is 0 if they point in the same direction, and $\pi$ if they point in opposite directions. The angle $\theta$ is the **angle between u and v**. Theorem 1 gives a formula to determine this angle.

**THEOREM 1 Angle Between Two Vectors**

The angle $\theta$ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}||\mathbf{v}|}\right).$$

Before proving Theorem 1 (which is a consequence of the law of cosines), let's focus attention on the expression $u_1 v_1 + u_2 v_2 + u_3 v_3$ in the calculation for $\theta$.

**DEFINITION Dot Product**

The **dot product** $\mathbf{u} \cdot \mathbf{v}$ ("**u** dot **v**") of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

**EXAMPLE 1** Finding Dot Products

**(a)** $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3)$
$= -6 - 4 + 3 = -7$

**(b)** $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$ ■

The dot product of a pair of two-dimensional vectors is defined in a similar fashion:

$$\langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2.$$

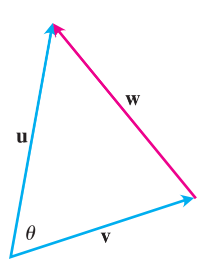



**FIGURE 12.20** The parallelogram law of addition of vectors gives $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

**Proof of Theorem 1** Applying the law of cosines (Equation (6), Section 1.6) to the triangle in Figure 12.20, we find that

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta \quad \text{Law of cosines}$$

$$2|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2.$$

Because $\mathbf{w} = \mathbf{u} - \mathbf{v}$, the component form of $\mathbf{w}$ is $\langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$. So

$$
\begin{aligned}
|\mathbf{u}|^2 &= \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = u_1^2 + u_2^2 + u_3^2 \\
|\mathbf{v}|^2 &= \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 = v_1^2 + v_2^2 + v_3^2 \\
|\mathbf{w}|^2 &= \left(\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}\right)^2 \\
&= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\
&= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2
\end{aligned}
$$

and

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Therefore,

$$
\begin{aligned}
2|\mathbf{u}||\mathbf{v}|\cos\theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3) \\
|\mathbf{u}||\mathbf{v}|\cos\theta &= u_1v_1 + u_2v_2 + u_3v_3 \\
\cos\theta &= \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}
\end{aligned}
$$

So

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right) \qquad \blacksquare$$

With the notation of the dot product, the angle between two vectors **u** and **v** can be written as

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right).$$

You Try It

## EXAMPLE 2 Finding the Angle Between Two Vectors in Space

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

**Solution** We use the formula above:

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4 \\
|\mathbf{u}| &= \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \\
|\mathbf{v}| &= \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7 \\
\theta &= \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) \\
&= \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians}. \qquad \blacksquare
\end{aligned}
$$

The angle formula applies to two-dimensional vectors as well.

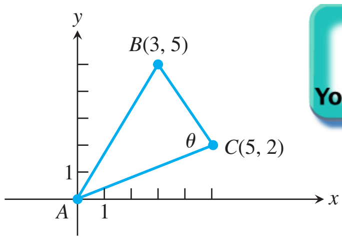



**FIGURE 12.21** The triangle in Example 3.

## EXAMPLE 3 Finding an Angle of a Triangle

Find the angle $\theta$ in the triangle $ABC$ determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ (Figure 12.21).

**Solution** The angle $\theta$ is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. The component forms of these two vectors are

$$\overrightarrow{CA} = \langle -5, -2 \rangle \qquad \text{and} \qquad \overrightarrow{CB} = \langle -2, 3 \rangle .$$

First we calculate the dot product and magnitudes of these two vectors.

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$
$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$
$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

Then applying the angle formula, we have

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}|\,|\overrightarrow{CB}|}\right) \\ &= \cos^{-1}\left(\frac{4}{\left(\sqrt{29}\right)\left(\sqrt{13}\right)}\right) \\ &\approx 78.1^\circ \qquad \text{or} \qquad 1.36 \text{ radians}. \end{aligned}$$

■

## Perpendicular (Orthogonal) Vectors

Two nonzero vectors **u** and **v** are perpendicular or **orthogonal** if the angle between them is $\pi/2$. For such vectors, we have $\mathbf{u} \cdot \mathbf{v} = 0$ because $\cos(\pi/2) = 0$. The converse is also true. If **u** and **v** are nonzero vectors with $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = 0$, then $\cos\theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$.

> **DEFINITION Orthogonal Vectors**
> Vectors **u** and **v** are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

## EXAMPLE 4 Applying the Definition of Orthogonality

**(a)** $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

**(b)** $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

**(c)** **0** is orthogonal to every vector **u** since

$$\begin{aligned} \mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) \\ &= 0. \end{aligned}$$

■

## Dot Product Properties and Vector Projections

The dot product obeys many of the laws that hold for ordinary products of real numbers (scalars).

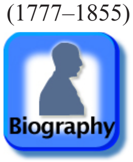
**Properties of the Dot Product**

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and $c$ is a scalar, then

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
2. $(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$
3. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$
4. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0}\cdot\mathbf{u} = 0.$

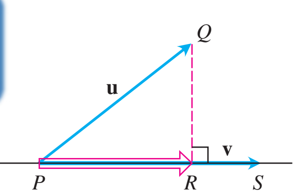
HISTORICAL BIOGRAPHY

Carl Friedrich Gauss
(1777–1855)




**Proofs of Properties 1 and 3** The properties are easy to prove using the definition. For instance, here are the proofs of Properties 1 and 3.

1. $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v}\cdot\mathbf{u}$
3. $$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \langle u_1, u_2, u_3\rangle\cdot\langle v_1+w_1, v_2+w_2, v_3+w_3\rangle \\ &= u_1(v_1+w_1) + u_2(v_2+w_2) + u_3(v_3+w_3) \\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + u_3v_3 + u_3w_3 \\ &= (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3) \\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}\end{aligned}$$ ■



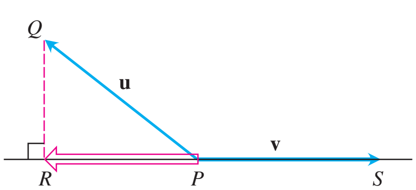




**FIGURE 12.22** The vector projection of **u** onto **v**.

We now return to the problem of projecting one vector onto another, posed in the opening to this section. The **vector projection** of $\mathbf{u} = \overrightarrow{PQ}$ onto a nonzero vector $\mathbf{v} = \overrightarrow{PS}$ (Figure 12.22) is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from $Q$ to the line $PS$. The notation for this vector is

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} \qquad (\text{"the vector projection of } \mathbf{u} \text{ onto } \mathbf{v}\text{"}).$$

If $\mathbf{u}$ represents a force, then $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ represents the effective force in the direction of $\mathbf{v}$ (Figure 12.23).

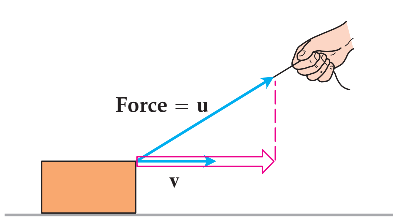



**FIGURE 12.23** If we pull on the box with force **u**, the effective force moving the box forward in the direction **v** is the projection of **u** onto **v**.

If the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$ is acute, $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ has length $|\mathbf{u}|\cos\theta$ and direction $\mathbf{v}/|\mathbf{v}|$ (Figure 12.24). If $\theta$ is obtuse, $\cos\theta < 0$ and $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ has length $-|\mathbf{u}|\cos\theta$ and direction $-\mathbf{v}/|\mathbf{v}|$. In both cases,

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= (|\mathbf{u}|\cos\theta)\frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} \qquad |\mathbf{u}|\cos\theta = \frac{|\mathbf{u}||\mathbf{v}|\cos\theta}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.\end{aligned}$$

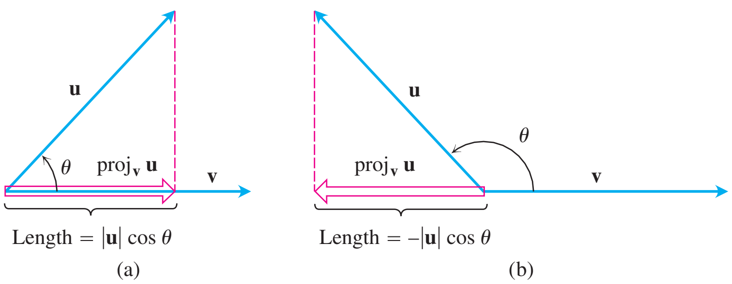



**FIGURE 12.24** The length of $\text{proj}_{\mathbf{v}}\mathbf{u}$ is (a) $|\mathbf{u}|\cos\theta$ if $\cos\theta \geq 0$ and (b) $-|\mathbf{u}|\cos\theta$ if $\cos\theta < 0$.

The number $|\mathbf{u}|\cos\theta$ is called the **scalar component of u in the direction of v**. To summarize,

Vector projection of **u** onto **v**:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \tag{1}$$

Scalar component of **u** in the direction of **v**:

$$|\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} \tag{2}$$

Note that both the vector projection of **u** onto **v** and the scalar component of **u** onto **v** depend only on the direction of the vector **v** and not its length (because we dot **u** with $\mathbf{v}/|\mathbf{v}|$, which is the direction of **v**).

## EXAMPLE 5 Finding the Vector Projection

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of **u** in the direction of **v**.

**Solution** We find $\text{proj}_{\mathbf{v}}\mathbf{u}$ from Equation (1):

$$\begin{aligned}\text{proj}_{\mathbf{v}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

We find the scalar component of **u** in the direction of **v** from Equation (2):

$$\begin{aligned}|\mathbf{u}|\cos\theta &= \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.\end{aligned}$$

Equations (1) and (2) also apply to two-dimensional vectors.

**EXAMPLE 6** Finding Vector Projections and Scalar Components

Find the vector projection of a force $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ and the scalar component of $\mathbf{F}$ in the direction of $\mathbf{v}$.

**Solution** The vector projection is

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\\ &= \frac{5-6}{1+9}(\mathbf{i}-3\mathbf{j}) = -\frac{1}{10}(\mathbf{i}-3\mathbf{j})\\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

The scalar component of $\mathbf{F}$ in the direction of $\mathbf{v}$ is

$$|\mathbf{F}|\cos\theta = \frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{5-6}{\sqrt{1+9}} = -\frac{1}{\sqrt{10}}.$$

## Work

In Chapter 6, we calculated the work done by a constant force of magnitude $F$ in moving an object through a distance $d$ as $W = Fd$. That formula holds only if the force is directed along the line of motion. If a force $\mathbf{F}$ moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ has some other direction, the work is performed by the component of $\mathbf{F}$ in the direction of $\mathbf{D}$. If $\theta$ is the angle between $\mathbf{F}$ and $\mathbf{D}$ (Figure 12.25), then

$$\begin{aligned}\text{Work} &= \begin{pmatrix}\text{scalar component of } \mathbf{F}\\ \text{in the direction of } \mathbf{D}\end{pmatrix}(\text{length of } \mathbf{D})\\ &= (|\mathbf{F}|\cos\theta)|\mathbf{D}|\\ &= \mathbf{F}\cdot\mathbf{D}.\end{aligned}$$

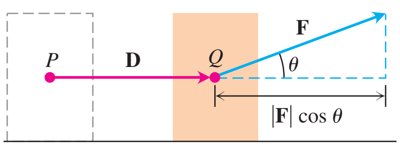



**FIGURE 12.25** The work done by a constant force $\mathbf{F}$ during a displacement $\mathbf{D}$ is $(|\mathbf{F}|\cos\theta)|\mathbf{D}|$.

**DEFINITION** **Work by Constant Force**

The **work** done by a constant force $\mathbf{F}$ acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta,$$

where $\theta$ is the angle between $\mathbf{F}$ and $\mathbf{D}$.

**EXAMPLE 7** Applying the Definition of Work

If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60°$, the work done by $\mathbf{F}$ in acting from $P$ to $Q$ is

$$\begin{aligned}\text{Work} &= |\mathbf{F}||\mathbf{D}|\cos\theta && \text{Definition}\\ &= (40)(3)\cos 60° && \text{Given values}\\ &= (120)(1/2)\\ &= 60 \text{ J (joules)}.\end{aligned}$$

We encounter more challenging work problems in Chapter 16 when we learn to find the work done by a variable force along a *path* in space.

## Writing a Vector as a Sum of Orthogonal Vectors

We know one way to write a vector $\mathbf{u} = \langle u_1, u_2 \rangle$ or $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ as a sum of two orthogonal vectors:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \qquad \text{or} \qquad \mathbf{u} = u_1\mathbf{i} + (u_2\mathbf{j} + u_3\mathbf{k})$$

(since $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$).

Sometimes, however, it is more informative to express $\mathbf{u}$ as a different sum. In mechanics, for instance, we often need to write a vector $\mathbf{u}$ as a sum of a vector parallel to a given vector $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$. As an example, in studying the motion of a particle moving along a path in the plane (or space), it is desirable to know the components of the acceleration vector in the direction of the tangent to the path (at a point) and of the normal to the path. (These *tangential* and *normal components* of acceleration are investigated in Section 13.4.) The acceleration vector can then be expressed as the sum of its (vector) tangential and normal components (which reflect important geometric properties about the nature of the path itself, such as *curvature*). Velocity and acceleration vectors are studied in the next chapter.

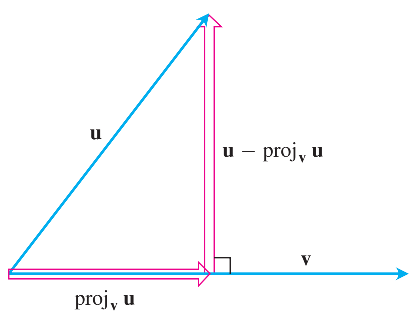



**FIGURE 12.26** Writing $\mathbf{u}$ as the sum of vectors parallel and orthogonal to $\mathbf{v}$.

Generally, for vectors $\mathbf{u}$ and $\mathbf{v}$, it is easy to see from Figure 12.26 that the vector

$$\mathbf{u} - \text{proj}_{\mathbf{v}}\, \mathbf{u}$$

is orthogonal to the projection vector $\text{proj}_{\mathbf{v}}\, \mathbf{u}$ (which has the same direction as $\mathbf{v}$). The following calculation verifies this observation:

$$\begin{aligned}
(\mathbf{u} - \text{proj}_{\mathbf{v}}\, \mathbf{u}) \cdot \text{proj}_{\mathbf{v}}\, \mathbf{u} &= \left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\right) \cdot \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} && \text{Equation (1)} \\
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)(\mathbf{u} \cdot \mathbf{v}) - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)^2 (\mathbf{v} \cdot \mathbf{v}) && \text{Dot product properties 2 and 3} \\
&= \frac{(\mathbf{u} \cdot \mathbf{v})^2}{|\mathbf{v}|^2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{|\mathbf{v}|^2} && \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \text{ cancels} \\
&= 0.
\end{aligned}$$

So the equation

$$\mathbf{u} = \text{proj}_{\mathbf{v}}\, \mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}}\, \mathbf{u})$$

expresses $\mathbf{u}$ as a sum of orthogonal vectors.

You Try It

### How to Write u as a Vector Parallel to v Plus a Vector Orthogonal to v

$$\mathbf{u} = \text{proj}_{\mathbf{v}}\, \mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}}\, \mathbf{u})$$

$$= \underbrace{\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}}_{\text{Parallel to } \mathbf{v}} + \underbrace{\left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\right)}_{\text{Orthogonal to } \mathbf{v}}$$

**EXAMPLE 8** Force on a Spacecraft

A force $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is applied to a spacecraft with velocity vector $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$. Express $\mathbf{F}$ as a sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

**Solution**

$$\begin{aligned}
\mathbf{F} &= \text{proj}_{\mathbf{v}}\,\mathbf{F} + (\mathbf{F} - \text{proj}_{\mathbf{v}}\,\mathbf{F}) \\
&= \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \left(\mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right) \\
&= \left(\frac{6-1}{9+1}\right)\mathbf{v} + \left(\mathbf{F} - \left(\frac{6-1}{9+1}\right)\mathbf{v}\right) \\
&= \frac{5}{10}(3\mathbf{i} - \mathbf{j}) + \left(2\mathbf{i} + \mathbf{j} - 3\mathbf{k} - \frac{5}{10}(3\mathbf{i} - \mathbf{j})\right) \\
&= \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right) + \left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right).
\end{aligned}$$

The force $(3/2)\mathbf{i} - (1/2)\mathbf{j}$ is the effective force parallel to the velocity $\mathbf{v}$. The force $(1/2)\mathbf{i} + (3/2)\mathbf{j} - 3\mathbf{k}$ is orthogonal to $\mathbf{v}$. To check that this vector is orthogonal to $\mathbf{v}$, we find the dot product:

$$\left(\frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right)\cdot(3\mathbf{i} - \mathbf{j}) = \frac{3}{2} - \frac{3}{2} = 0.$$

■