EXERCISES 12.3

Dot Product and Projections

In Exercises 1–8, find

a. $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$

b. the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$

c. the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$

d. the vector $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

1. $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \sqrt{5}\mathbf{k}, \quad \mathbf{u} = -2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$

2. $\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{k}, \quad \mathbf{u} = 5\mathbf{i} + 12\mathbf{j}$

3. $\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}, \quad \mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$

4. $\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}, \quad \mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

5. $\mathbf{v} = 5\mathbf{j} - 3\mathbf{k}, \quad \mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

6. $\mathbf{v} = -\mathbf{i} + \mathbf{j}, \quad \mathbf{u} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}$

7. $\mathbf{v} = 5\mathbf{i} + \mathbf{j}, \quad \mathbf{u} = 2\mathbf{i} + \sqrt{17}\mathbf{j}$

8. $\mathbf{v} = \left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{3}} \right\rangle, \quad \mathbf{u} = \left\langle \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{3}} \right\rangle$

T Angles Between Vectors

Find the angles between the vectors in Exercises 9–12 to the nearest hundredth of a radian.

9. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

10. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$

11. $\mathbf{u} = \sqrt{3}\mathbf{i} - 7\mathbf{j}, \quad \mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

12. $\mathbf{u} = \mathbf{i} + \sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}, \quad \mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$

13. Triangle Find the measures of the angles of the triangle whose vertices are $A = (-1, 0)$, $B = (2, 1)$, and $C = (1, -2)$.

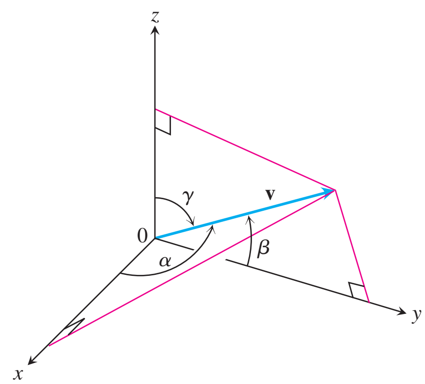
14. Rectangle Find the measures of the angles between the diagonals of the rectangle whose vertices are $A = (1, 0)$, $B = (0, 3)$, $C = (3, 4)$, and $D = (4, 1)$.

15. Direction angles and direction cosines The *direction angles* α, β, and γ of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are defined as follows:

α is the angle between $\mathbf{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$

β is the angle between $\mathbf{v}$ and the positive y-axis $(0 \le \beta \le \pi)$

γ is the angle between $\mathbf{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$.



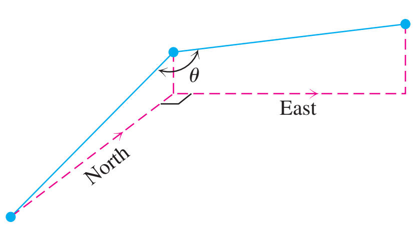
a. Show that

$$\cos \alpha = \frac{a}{|\mathbf{v}|}, \qquad \cos \beta = \frac{b}{|\mathbf{v}|}, \qquad \cos \gamma = \frac{c}{|\mathbf{v}|},$$

and $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. These cosines are called the *direction cosines* of $\mathbf{v}$.

b. **Unit vectors are built from direction cosines** Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector, then a, b, and c are the direction cosines of $\mathbf{v}$.

16. **Water main construction** A water main is to be constructed with a 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.



Decomposing Vectors

In Exercises 17–19, write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

17. $\mathbf{u} = 3\mathbf{j} + 4\mathbf{k}, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}$

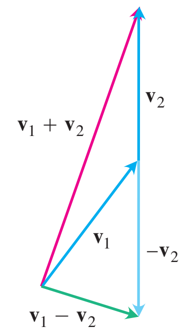
18. $\mathbf{u} = \mathbf{j} + \mathbf{k}, \quad \mathbf{v} = \mathbf{i} + \mathbf{j}$

19. $\mathbf{u} = 8\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

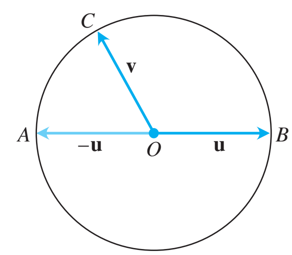
20. **Sum of vectors** $\mathbf{u} = \mathbf{i} + (\mathbf{j} + \mathbf{k})$ is already the sum of a vector parallel to $\mathbf{i}$ and a vector orthogonal to $\mathbf{i}$. If you use $\mathbf{v} = \mathbf{i}$, in the decomposition $\mathbf{u} = \text{proj}_{\mathbf{v}}\,\mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u})$, do you get $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \mathbf{i}$ and $(\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}) = \mathbf{j} + \mathbf{k}$? Try it and find out.

Geometry and Examples

21. **Sums and differences** In the accompanying figure, it looks as if $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are orthogonal. Is this mere coincidence, or are there circumstances under which we may expect the sum of two vectors to be orthogonal to their difference? Give reasons for your answer.



22. **Orthogonality on a circle** Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B. Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

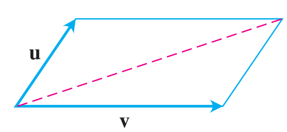


23. **Diagonals of a rhombus** Show that the diagonals of a rhombus (parallelogram with sides of equal length) are perpendicular.

24. **Perpendicular diagonals** Show that squares are the only rectangles with perpendicular diagonals.

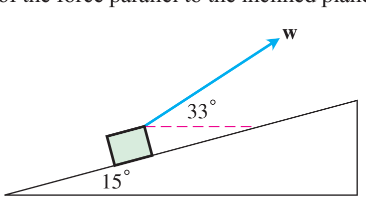
25. **When parallelograms are rectangles** Prove that a parallelogram is a rectangle if and only if its diagonals are equal in length. (This fact is often exploited by carpenters.)

26. **Diagonal of parallelogram** Show that the indicated diagonal of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$ if $|\mathbf{u}| = |\mathbf{v}|$.



27. **Projectile motion** A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

28. **Inclined plane** Suppose that a box is being towed up an inclined plane as shown in the figure. Find the force $\mathbf{w}$ needed to make the component of the force parallel to the inclined plane equal to 2.5 lb.

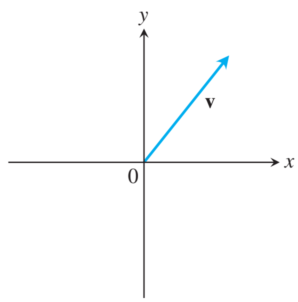


Theory and Examples

29. a. Cauchy-Schwartz inequality Use the fact that $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ to show that the inequality $|\mathbf{u}\cdot\mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$ holds for any vectors $\mathbf{u}$ and $\mathbf{v}$.

b. Under what circumstances, if any, does $|\mathbf{u}\cdot\mathbf{v}|$ equal $|\mathbf{u}||\mathbf{v}|$? Give reasons for your answer.

30. Copy the axes and vector shown here. Then shade in the points (x, y) for which $(x\mathbf{i} + y\mathbf{j})\cdot\mathbf{v} \le 0$. Justify your answer.



31. Orthogonal unit vectors If $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal unit vectors and $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2$, find $\mathbf{v}\cdot\mathbf{u}_1$.

32. Cancellation in dot products In real-number multiplication, if $uv_1 = uv_2$ and $u \ne 0$, we can cancel the u and conclude that $v_1 = v_2$. Does the same rule hold for the dot product: If $\mathbf{u}\cdot\mathbf{v}_1 = \mathbf{u}\cdot\mathbf{v}_2$ and $\mathbf{u} \ne \mathbf{0}$, can you conclude that $\mathbf{v}_1 = \mathbf{v}_2$? Give reasons for your answer.

Equations for Lines in the Plane

33. Line perpendicular to a vector Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$ by establishing that the slope of $\mathbf{v}$ is the negative reciprocal of the slope of the given line.

34. Line parallel to a vector Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is parallel to the line $bx - ay = c$ by establishing that the slope of the line segment representing $\mathbf{v}$ is the same as the slope of the given line.

In Exercises 35–38, use the result of Exercise 33 to find an equation for the line through P perpendicular to $\mathbf{v}$. Then sketch the line. Include $\mathbf{v}$ in your sketch *as a vector starting at the origin.*

35. $P(2, 1)$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

36. $P(-1, 2)$, $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$

37. $P(-2, -7)$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$

38. $P(11, 10)$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

In Exercises 39–42, use the result of Exercise 34 to find an equation for the line through P parallel to $\mathbf{v}$. Then sketch the line. Include $\mathbf{v}$ in your sketch *as a vector starting at the origin.*

39. $P(-2, 1)$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

40. $P(0, -2)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

41. $P(1, 2)$, $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$

42. $P(1, 3)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

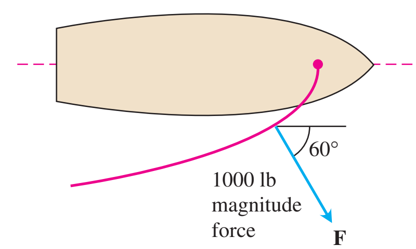
Work

43. Work along a line Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters).

44. Locomotive The union Pacific's *Big Boy* locomotive could pull 6000-ton trains with a tractive effort (pull) of 602,148 N (135,375 lb). At this level of effort, about how much work did *Big Boy* do on the (approximately straight) 605-km journey from San Francisco to Los Angeles?

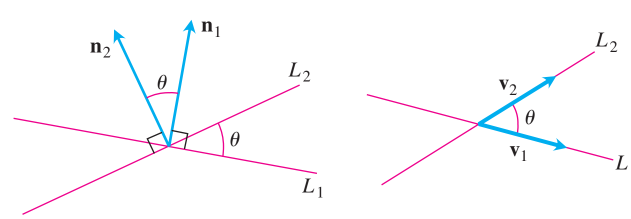
45. Inclined plane How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?

46. Sailboat The wind passing over a boat's sail exerted a 1000-lb magnitude force $\mathbf{F}$ as shown here. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.



Angles Between Lines in the Plane

The acute angle between intersecting lines that do not cross at right angles is the same as the angle determined by vectors normal to the lines or by the vectors parallel to the lines.



Use this fact and the results of Exercise 33 or 34 to find the acute angles between the lines in Exercises 47–52.

47. $3x + y = 5$, $2x - y = 4$

48. $y = \sqrt{3}x - 1$, $y = -\sqrt{3}x + 2$

49. $\sqrt{3}x - y = -2$, $x - \sqrt{3}y = 1$

50. $x + \sqrt{3}y = 1$, $\left(1 - \sqrt{3}\right)x + \left(1 + \sqrt{3}\right)y = 8$

51. $3x - 4y = 3$, $x - y = 7$

52. $12x + 5y = 1$, $2x - 2y = 3$

Angles Between Differentiable Curves

The angles between two differentiable curves at a point of intersection are the angles between the curves' tangent lines at these points. Find the angles between the curves in Exercises 53–56. Note that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is a vector in the plane, then the vector has slope b/a provided $a \neq 0$.

53. $y = (3/2) - x^2, \quad y = x^2$ (two points of intersection)

54. $x = (3/4) - y^2, \quad x = y^2 - (3/4)$ (two points of intersection)

55. $y = x^3, \quad x = y^2$ (two points of intersection)

56. $y = -x^2, \quad y = \sqrt{x}$ (two points of intersection)