

13.2

Modeling Projectile Motion

When we shoot a projectile into the air we usually want to know beforehand how far it will go (will it reach the target?), how high it will rise (will it clear the hill?), and when it will land (when do we get results?). We get this information from the direction and magnitude of the projectile's initial velocity vector, using Newton's second law of motion.

The Vector and Parametric Equations for Ideal Projectile Motion

To derive equations for projectile motion, we assume that the projectile behaves like a particle moving in a vertical coordinate plane and that the only force acting on the projectile during its flight is the constant force of gravity, which always points straight down. In practice, none of these assumptions really holds. The ground moves beneath the projectile as the earth turns, the air creates a frictional force that varies with the projectile's speed and altitude, and the force of gravity changes as the projectile moves along. All this must be taken into account by applying corrections to the predictions of the *ideal* equations we are about to derive. The corrections, however, are not the subject of this section.

We assume that the projectile is launched from the origin at time $t = 0$ into the first quadrant with an initial velocity \mathbf{v}_0 (Figure 13.9). If \mathbf{v}_0 makes an angle α with the horizontal, then

$$\mathbf{v}_0 = (|\mathbf{v}_0| \cos \alpha)\mathbf{i} + (|\mathbf{v}_0| \sin \alpha)\mathbf{j}.$$

If we use the simpler notation v_0 for the initial speed $|\mathbf{v}_0|$, then

$$\mathbf{v}_0 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}. \quad (1)$$

The projectile's initial position is

$$\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}. \quad (2)$$

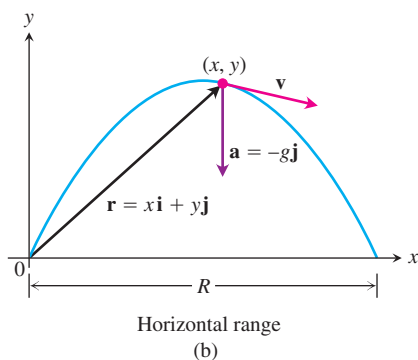
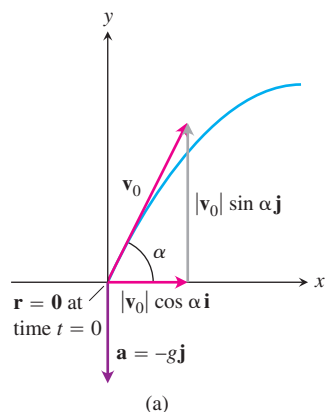


FIGURE 13.9 (a) Position, velocity, acceleration, and launch angle at $t = 0$. (b) Position, velocity, and acceleration at a later time t .

Newton's second law of motion says that the force acting on the projectile is equal to the projectile's mass m times its acceleration, or $m(d^2\mathbf{r}/dt^2)$ if \mathbf{r} is the projectile's position vector and t is time. If the force is solely the gravitational force $-mg\mathbf{j}$, then

$$m \frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{j} \quad \text{and} \quad \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j}.$$

We find \mathbf{r} as a function of t by solving the following initial value problem.

$$\text{Differential equation:} \quad \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j}$$

$$\text{Initial conditions:} \quad \mathbf{r} = \mathbf{r}_0 \quad \text{and} \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}_0 \quad \text{when } t = 0$$

The first integration gives

$$\frac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0.$$

A second integration gives

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0t + \mathbf{r}_0.$$

Substituting the values of \mathbf{v}_0 and \mathbf{r}_0 from Equations (1) and (2) gives

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \underbrace{(v_0 \cos \alpha)t\mathbf{i} + (v_0 \sin \alpha)t\mathbf{j}}_{\mathbf{v}_0t} + \mathbf{0}$$

Collecting terms, we have

Ideal Projectile Motion Equation

$$\mathbf{r} = (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j}. \quad (3)$$

Equation (3) is the *vector equation* for ideal projectile motion. The angle α is the projectile's **launch angle (firing angle, angle of elevation)**, and v_0 , as we said before, is the projectile's **initial speed**. The components of \mathbf{r} give the parametric equations

$$x = (v_0 \cos \alpha)t \quad \text{and} \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2, \quad (4)$$

where x is the distance downrange and y is the height of the projectile at time $t \geq 0$.

EXAMPLE 1 Firing an Ideal Projectile

A projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60° . Where will the projectile be 10 sec later?

Solution We use Equation (3) with $v_0 = 500$, $\alpha = 60^\circ$, $g = 9.8$, and $t = 10$ to find the projectile's components 10 sec after firing.

$$\begin{aligned}\mathbf{r} &= (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j} \\ &= (500)\left(\frac{1}{2}\right)(10)\mathbf{i} + \left((500)\left(\frac{\sqrt{3}}{2}\right)10 - \left(\frac{1}{2}\right)(9.8)(100) \right)\mathbf{j} \\ &\approx 2500\mathbf{i} + 3840\mathbf{j}.\end{aligned}$$

Ten seconds after firing, the projectile is about 3840 m in the air and 2500 m downrange. ■

Height, Flight Time, and Range

Equation (3) enables us to answer most questions about the ideal motion for a projectile fired from the origin.

The projectile reaches its highest point when its vertical velocity component is zero, that is, when

$$\frac{dy}{dt} = v_0 \sin \alpha - gt = 0, \quad \text{or} \quad t = \frac{v_0 \sin \alpha}{g}.$$

For this value of t , the value of y is

$$y_{\max} = (v_0 \sin \alpha)\left(\frac{v_0 \sin \alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin \alpha}{g}\right)^2 = \frac{(v_0 \sin \alpha)^2}{2g}.$$

To find when the projectile lands when fired over horizontal ground, we set the vertical component equal to zero in Equation (3) and solve for t .

$$\begin{aligned}(v_0 \sin \alpha)t - \frac{1}{2}gt^2 &= 0 \\ t\left(v_0 \sin \alpha - \frac{1}{2}gt\right) &= 0 \\ t = 0, \quad t &= \frac{2v_0 \sin \alpha}{g}\end{aligned}$$

Since 0 is the time the projectile is fired, $(2v_0 \sin \alpha)/g$ must be the time when the projectile strikes the ground.

To find the projectile's **range** R , the distance from the origin to the point of impact on horizontal ground, we find the value of the horizontal component when $t = (2v_0 \sin \alpha)/g$.

$$\begin{aligned}x &= (v_0 \cos \alpha)t \\ R &= (v_0 \cos \alpha)\left(\frac{2v_0 \sin \alpha}{g}\right) = \frac{v_0^2}{g}(2 \sin \alpha \cos \alpha) = \frac{v_0^2}{g} \sin 2\alpha\end{aligned}$$

The range is largest when $\sin 2\alpha = 1$ or $\alpha = 45^\circ$.

Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α :

$$\text{Maximum height: } y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

$$\text{Flight time: } t = \frac{2v_0 \sin \alpha}{g}$$

$$\text{Range: } R = \frac{v_0^2}{g} \sin 2\alpha.$$

EXAMPLE 2 Investigating Ideal Projectile Motion

Find the maximum height, flight time, and range of a projectile fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60° (same projectile as Example 1).

Solution

$$\begin{aligned} \text{Maximum height: } y_{\max} &= \frac{(v_0 \sin \alpha)^2}{2g} \\ &= \frac{(500 \sin 60^\circ)^2}{2(9.8)} \approx 9566 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Flight time: } t &= \frac{2v_0 \sin \alpha}{g} \\ &= \frac{2(500) \sin 60^\circ}{9.8} \approx 88.4 \text{ sec} \end{aligned}$$

$$\begin{aligned} \text{Range: } R &= \frac{v_0^2}{g} \sin 2\alpha \\ &= \frac{(500)^2 \sin 120^\circ}{9.8} \approx 22,092 \text{ m} \end{aligned}$$

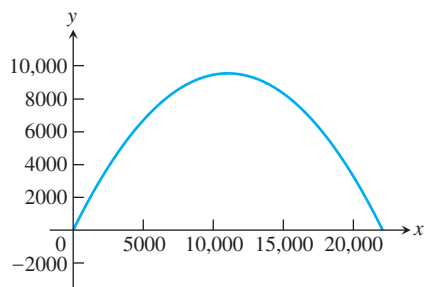


FIGURE 13.10 The graph of the projectile described in Example 2.

From Equation (3), the position vector of the projectile is

$$\begin{aligned} \mathbf{r} &= (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j} \\ &= (500 \cos 60^\circ)t\mathbf{i} + \left((500 \sin 60^\circ)t - \frac{1}{2}(9.8)t^2 \right)\mathbf{j} \\ &= 250t\mathbf{i} + \left((250\sqrt{3})t - 4.9t^2 \right)\mathbf{j}. \end{aligned}$$

A graph of the projectile's path is shown in Figure 13.10. ■

Ideal Trajectories Are Parabolic

It is often claimed that water from a hose traces a parabola in the air, but anyone who looks closely enough will see this is not so. The air slows the water down, and its forward progress is too slow at the end to keep pace with the rate at which it falls.

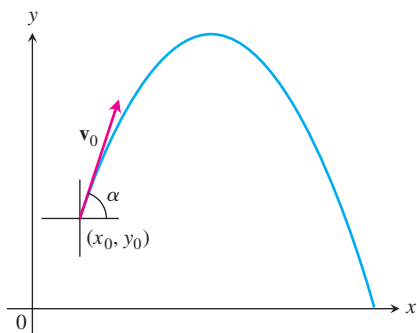


FIGURE 13.11 The path of a projectile fired from (x_0, y_0) with an initial velocity v_0 at an angle of α degrees with the horizontal.

What is really being claimed is that ideal projectiles move along parabolas, and this we can see from Equations (4). If we substitute $t = x/(v_0 \cos \alpha)$ from the first equation into the second, we obtain the Cartesian-coordinate equation

$$y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x.$$

This equation has the form $y = ax^2 + bx$, so its graph is a parabola.

Firing from (x_0, y_0)

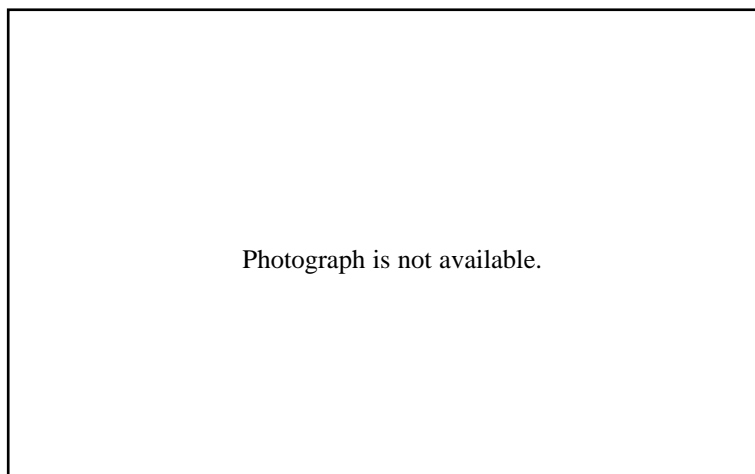
If we fire our ideal projectile from the point (x_0, y_0) instead of the origin (Figure 13.11), the position vector for the path of motion is

$$\mathbf{r} = (x_0 + (v_0 \cos \alpha)t)\mathbf{i} + \left(y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}, \quad (5)$$

as you are asked to show in Exercise 19.

EXAMPLE 3 Firing a Flaming Arrow

To open the 1992 Summer Olympics in Barcelona, bronze medalist archer Antonio Rebollo lit the Olympic torch with a flaming arrow (Figure 13.12). Suppose that Rebollo shot the arrow at a height of 6 ft above ground level 90 ft from the 70-ft-high cauldron, and he wanted the arrow to reach maximum height exactly 4 ft above the center of the cauldron (Figure 13.12).



Photograph is not available.

FIGURE 13.12 Spanish archer Antonio Rebollo lights the Olympic torch in Barcelona with a flaming arrow.

- Express y_{\max} in terms of the initial speed v_0 and firing angle α .
- Use $y_{\max} = 74$ ft (Figure 13.13) and the result from part (a) to find the value of $v_0 \sin \alpha$.
- Find the value of $v_0 \cos \alpha$.
- Find the initial firing angle of the arrow.

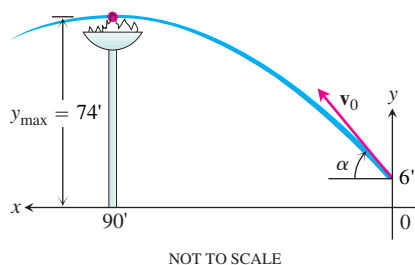


FIGURE 13.13 Ideal path of the arrow that lit the Olympic torch (Example 3).

Solution

- (a) We use a coordinate system in which the positive x -axis lies along the ground toward the left (to match the second photograph in Figure 13.12) and the coordinates of the flaming arrow at $t = 0$ are $x_0 = 0$ and $y_0 = 6$ (Figure 13.13). We have

$$y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \quad \text{Equation (5), j-component}$$

$$= 6 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2. \quad y_0 = 6$$

We find the time when the arrow reaches its highest point by setting $dy/dt = 0$ and solving for t , obtaining

$$t = \frac{v_0 \sin \alpha}{g}.$$

For this value of t , the value of y is

$$\begin{aligned} y_{\max} &= 6 + (v_0 \sin \alpha) \left(\frac{v_0 \sin \alpha}{g} \right) - \frac{1}{2}g \left(\frac{v_0 \sin \alpha}{g} \right)^2 \\ &= 6 + \frac{(v_0 \sin \alpha)^2}{2g}. \end{aligned}$$

- (b) Using $y_{\max} = 74$ and $g = 32$, we see from the preceding equation in part (a) that

$$74 = 6 + \frac{(v_0 \sin \alpha)^2}{2(32)}$$

or

$$v_0 \sin \alpha = \sqrt{(68)(64)}.$$

- (c) When the arrow reaches y_{\max} , the horizontal distance traveled to the center of the cauldron is $x = 90$ ft. We substitute the time to reach y_{\max} from part (a) and the horizontal distance $x = 90$ ft into the i -component of Equation (5) to obtain

$$x = x_0 + (v_0 \cos \alpha)t \quad \text{Equation (5), i-component}$$

$$90 = 0 + (v_0 \cos \alpha)t \quad x = 90, x_0 = 0$$

$$= (v_0 \cos \alpha) \left(\frac{v_0 \sin \alpha}{g} \right). \quad t = (v_0 \sin \alpha)/g$$

Solving this equation for $v_0 \cos \alpha$ and using $g = 32$ and the result from part (b), we have

$$v_0 \cos \alpha = \frac{90g}{v_0 \sin \alpha} = \frac{(90)(32)}{\sqrt{(68)(64)}}.$$

- (d) Parts (b) and (c) together tell us that

$$\tan \alpha = \frac{v_0 \sin \alpha}{v_0 \cos \alpha} = \frac{(\sqrt{(68)(64)})^2}{(90)(32)} = \frac{68}{45}$$

or

$$\alpha = \tan^{-1}\left(\frac{68}{45}\right) \approx 56.5^\circ.$$

This is Rebollo's firing angle. ■

Projectile Motion with Wind Gusts

The next example shows how to account for another force acting on a projectile. We also assume that the path of the baseball in Example 4 lies in a vertical plane.

EXAMPLE 4 Hitting a Baseball

A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/sec, making an angle of 20° with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-8.8\mathbf{i}$ (ft/sec) to the ball's initial velocity (8.8 ft/sec = 6 mph).

- (a) Find a vector equation (position vector) for the path of the baseball.
- (b) How high does the baseball go, and when does it reach maximum height?
- (c) Assuming that the ball is not caught, find its range and flight time.

Solution

- (a) Using Equation (1) and accounting for the gust of wind, the initial velocity of the baseball is

$$\begin{aligned} \mathbf{v}_0 &= (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} - 8.8\mathbf{i} \\ &= (152 \cos 20^\circ)\mathbf{i} + (152 \sin 20^\circ)\mathbf{j} - (8.8)\mathbf{i} \\ &= (152 \cos 20^\circ - 8.8)\mathbf{i} + (152 \sin 20^\circ)\mathbf{j}. \end{aligned}$$

The initial position is $\mathbf{r}_0 = 0\mathbf{i} + 3\mathbf{j}$. Integration of $d^2\mathbf{r}/dt^2 = -g\mathbf{j}$ gives

$$\frac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0.$$

A second integration gives

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0t + \mathbf{r}_0.$$

Substituting the values of \mathbf{v}_0 and \mathbf{r}_0 into the last equation gives the position vector of the baseball.

$$\begin{aligned} \mathbf{r} &= -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0t + \mathbf{r}_0 \\ &= -16t^2\mathbf{j} + (152 \cos 20^\circ - 8.8)t\mathbf{i} + (152 \sin 20^\circ)t\mathbf{j} + 3\mathbf{j} \\ &= (152 \cos 20^\circ - 8.8)t\mathbf{i} + \left(3 + (152 \sin 20^\circ)t - 16t^2\right)\mathbf{j}. \end{aligned}$$

- (b) The baseball reaches its highest point when the vertical component of velocity is zero, or

$$\frac{dy}{dt} = 152 \sin 20^\circ - 32t = 0.$$

Solving for t we find

$$t = \frac{152 \sin 20^\circ}{32} \approx 1.62 \text{ sec.}$$

Substituting this time into the vertical component for \mathbf{r} gives the maximum height

$$\begin{aligned} y_{\max} &= 3 + (152 \sin 20^\circ)(1.62) - 16(1.62)^2 \\ &\approx 45.2 \text{ ft.} \end{aligned}$$

That is, the maximum height of the baseball is about 45.2 ft, reached about 1.6 sec after leaving the bat.

- (c) To find when the baseball lands, we set the vertical component for \mathbf{r} equal to 0 and solve for t :

$$\begin{aligned} 3 + (152 \sin 20^\circ)t - 16t^2 &= 0 \\ 3 + (51.99)t - 16t^2 &= 0. \end{aligned}$$

The solution values are about $t = 3.3$ sec and $t = -0.06$ sec. Substituting the positive time into the horizontal component for \mathbf{r} , we find the range

$$\begin{aligned} R &= (152 \cos 20^\circ - 8.8)(3.3) \\ &\approx 442 \text{ ft.} \end{aligned}$$

Thus, the horizontal range is about 442 ft, and the flight time is about 3.3 sec. ■

In Exercises 29 through 31, we consider projectile motion when there is air resistance slowing down the flight.