

13.5

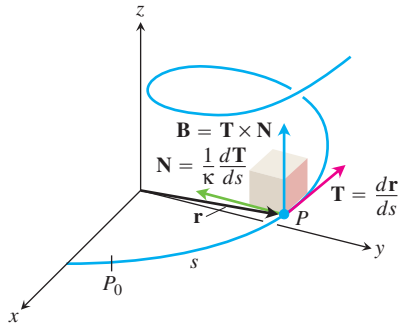
Torsion and the Unit Binormal Vector **B**

FIGURE 13.25 The **TNB** frame of mutually orthogonal unit vectors traveling along a curve in space.

If you are traveling along a space curve, the Cartesian **i**, **j**, and **k** coordinate system for representing the vectors describing your motion are not truly relevant to you. What is meaningful instead are the vectors representative of your forward direction (the unit tangent vector **T**), the direction in which your path is turning (the unit normal vector **N**), and the tendency of your motion to “twist” out of the plane created by these vectors in the direction perpendicular to this plane (defined by the *unit binormal vector* **B** = **T** × **N**). Expressing the acceleration vector along the curve as a linear combination of this **TNB** frame of mutually orthogonal unit vectors traveling with the motion (Figure 13.25) is particularly revealing of the nature of the path and motion along it.

Torsion

The **binormal vector** of a curve in space is **B** = **T** × **N**, a unit vector orthogonal to both **T** and **N** (Figure 13.26). Together **T**, **N**, and **B** define a moving right-handed vector frame that plays a significant role in calculating the paths of particles moving through space. It is

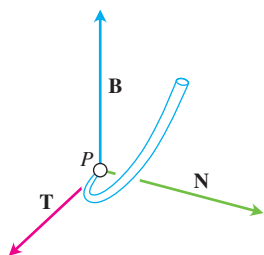


FIGURE 13.26 The vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} (in that order) make a right-handed frame of mutually orthogonal unit vectors in space.

called the **Frenet** (“fre-nay”) **frame** (after Jean-Frédéric Frenet, 1816–1900), or the **TNB frame**.

How does $d\mathbf{B}/ds$ behave in relation to \mathbf{T} , \mathbf{N} , and \mathbf{B} ? From the rule for differentiating a cross product, we have

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}.$$

Since \mathbf{N} is the direction of $d\mathbf{T}/ds$, $(d\mathbf{T}/ds) \times \mathbf{N} = \mathbf{0}$ and

$$\frac{d\mathbf{B}}{ds} = \mathbf{0} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}.$$

From this we see that $d\mathbf{B}/ds$ is orthogonal to \mathbf{T} since a cross product is orthogonal to its factors.

Since $d\mathbf{B}/ds$ is also orthogonal to \mathbf{B} (the latter has constant length), it follows that $d\mathbf{B}/ds$ is orthogonal to the plane of \mathbf{B} and \mathbf{T} . In other words, $d\mathbf{B}/ds$ is parallel to \mathbf{N} , so $d\mathbf{B}/ds$ is a scalar multiple of \mathbf{N} . In symbols,

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

The negative sign in this equation is traditional. The scalar τ is called the *torsion* along the curve. Notice that

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\mathbf{N} \cdot \mathbf{N} = -\tau(1) = -\tau,$$

so that

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$

DEFINITION Torsion

Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion** function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}. \quad (1)$$

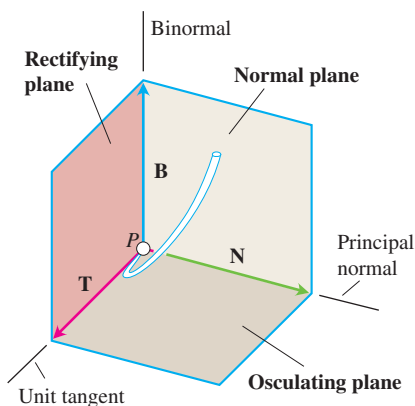


FIGURE 13.27 The names of the three planes determined by \mathbf{T} , \mathbf{N} , and \mathbf{B} .

Unlike the curvature κ , which is never negative, the torsion τ may be positive, negative, or zero.

The three planes determined by \mathbf{T} , \mathbf{N} , and \mathbf{B} are named and shown in Figure 13.27. The curvature $\kappa = |d\mathbf{T}/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. Similarly, the torsion $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ is the rate at which the osculating plane turns about \mathbf{T} as P moves along the curve. Torsion measures how the curve twists.

If we think of the curve as the path of a moving body, then $|d\mathbf{T}/ds|$ tells how much the path turns to the left or right as the object moves along; it is called the *curvature* of the object’s path. The number $-(d\mathbf{B}/ds) \cdot \mathbf{N}$ tells how much a body’s path rotates or

twists out of its plane of motion as the object moves along; it is called the *torsion* of the body's path. Look at Figure 13.28. If P is a train climbing up a curved track, the rate at which the headlight turns from side to side per unit distance is the curvature of the track. The rate at which the engine tends to twist out of the plane formed by \mathbf{T} and \mathbf{N} is the torsion.

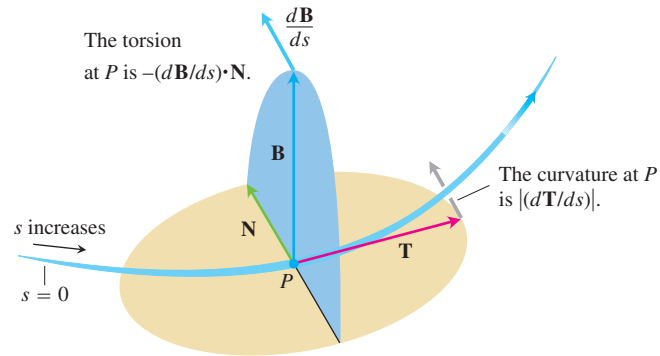


FIGURE 13.28 Every moving body travels with a **TNB** frame that characterizes the geometry of its path of motion.

Tangential and Normal Components of Acceleration

When a body is accelerated by gravity, brakes, a combination of rocket motors, or whatever, we usually want to know how much of the acceleration acts in the direction of motion, in the tangential direction \mathbf{T} . We can calculate this using the Chain Rule to rewrite \mathbf{v} as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt} = \mathbf{T} \frac{ds}{dt}$$

and differentiating both ends of this string of equalities to get

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left(\mathbf{T} \frac{ds}{dt} \right) = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \frac{d\mathbf{T}}{dt} \\ &= \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \left(\frac{d\mathbf{T}}{ds} \frac{ds}{dt} \right) = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \left(\kappa \mathbf{N} \frac{ds}{dt} \right) \quad \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \\ &= \frac{d^2s}{dt^2} \mathbf{T} + \kappa \left(\frac{ds}{dt} \right)^2 \mathbf{N}. \end{aligned}$$

DEFINITION Tangential and Normal Components of Acceleration

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}, \quad (2)$$

where

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt} |\mathbf{v}| \quad \text{and} \quad a_N = \kappa \left(\frac{ds}{dt} \right)^2 = \kappa |\mathbf{v}|^2 \quad (3)$$

are the **tangential** and **normal** scalar components of acceleration.

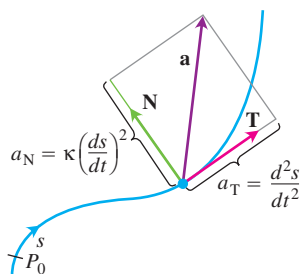


FIGURE 13.29 The tangential and normal components of acceleration. The acceleration \mathbf{a} always lies in the plane of \mathbf{T} and \mathbf{N} , orthogonal to \mathbf{B} .

Notice that the binormal vector \mathbf{B} does not appear in Equation (2). No matter how the path of the moving body we are watching may appear to twist and turn in space, the acceleration \mathbf{a} always lies in the plane of \mathbf{T} and \mathbf{N} orthogonal to \mathbf{B} . The equation also tells us exactly how much of the acceleration takes place tangent to the motion (d^2s/dt^2) and how much takes place normal to the motion [$\kappa(ds/dt)^2$] (Figure 13.29).

What information can we glean from Equations (3)? By definition, acceleration \mathbf{a} is the rate of change of velocity \mathbf{v} , and in general, both the length and direction of \mathbf{v} change as a body moves along its path. The tangential component of acceleration a_T measures the rate of change of the *length* of \mathbf{v} (that is, the change in the speed). The normal component of acceleration a_N measures the rate of change of the *direction* of \mathbf{v} .

Notice that the normal scalar component of the acceleration is the curvature times the *square* of the speed. This explains why you have to hold on when your car makes a sharp (large κ), high-speed (large $|\mathbf{v}|$) turn. If you double the speed of your car, you will experience four times the normal component of acceleration for the same curvature.

If a body moves in a circle at a constant speed, d^2s/dt^2 is zero and all the acceleration points along \mathbf{N} toward the circle's center. If the body is speeding up or slowing down, \mathbf{a} has a nonzero tangential component (Figure 13.30).

To calculate a_N , we usually use the formula $a_N = \sqrt{|\mathbf{a}|^2 - a_T^2}$, which comes from solving the equation $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_T^2 + a_N^2$ for a_N . With this formula, we can find a_N without having to calculate κ first.

Formula for Calculating the Normal Component of Acceleration

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} \quad (4)$$

EXAMPLE 1 Finding the Acceleration Scalar Components a_T , a_N

Without finding \mathbf{T} and \mathbf{N} , write the acceleration of the motion

$$\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad t > 0$$

in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$. (The path of the motion is the involute of the circle in Figure 13.31.)

Solution We use the first of Equations (3) to find a_T :

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = (-\sin t + \sin t + t \cos t)\mathbf{i} + (\cos t - \cos t + t \sin t)\mathbf{j} \\ &= (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \end{aligned}$$

$$|\mathbf{v}| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} = \sqrt{t^2} = |t| = t \quad t > 0$$

$$a_T = \frac{d}{dt} |\mathbf{v}| = \frac{d}{dt} (t) = 1. \quad \text{Equation (3)}$$

Knowing a_T , we use Equation (4) to find a_N :

$$\begin{aligned} \mathbf{a} &= (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j} \\ |\mathbf{a}|^2 &= t^2 + 1 && \text{After some algebra} \\ a_N &= \sqrt{|\mathbf{a}|^2 - a_T^2} \\ &= \sqrt{(t^2 + 1) - (1)} = \sqrt{t^2} = t. \end{aligned}$$

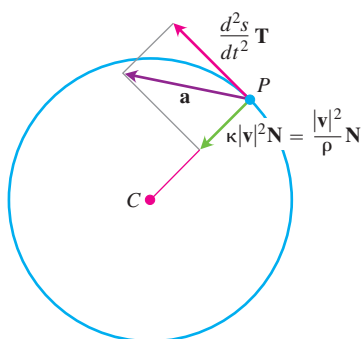


FIGURE 13.30 The tangential and normal components of the acceleration of a body that is speeding up as it moves counterclockwise around a circle of radius ρ .

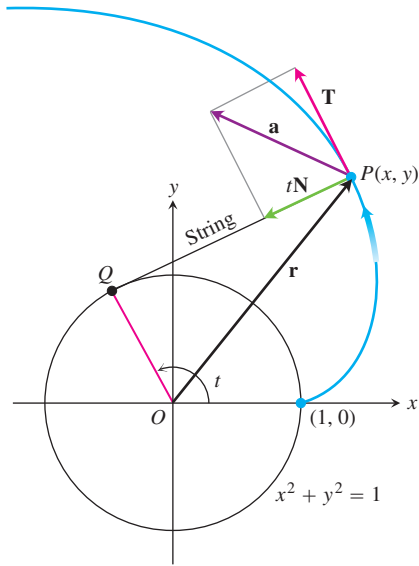


FIGURE 13.31 The tangential and normal components of the acceleration of the motion $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}$, for $t > 0$. If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end P traces an involute of the circle (Example 1).

We then use Equation (2) to find \mathbf{a} :

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$

Formulas for Computing Curvature and Torsion

We now give some easy-to-use formulas for computing the curvature and torsion of a smooth curve. From Equation (2), we have

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \left(\frac{ds}{dt} \mathbf{T} \right) \times \left[\frac{d^2s}{dt^2} \mathbf{T} + \kappa \left(\frac{ds}{dt} \right)^2 \mathbf{N} \right] & \mathbf{v} = d\mathbf{r}/dt = (ds/dt)\mathbf{T} \\ &= \left(\frac{ds}{dt} \frac{d^2s}{dt^2} \right) (\mathbf{T} \times \mathbf{T}) + \kappa \left(\frac{ds}{dt} \right)^3 (\mathbf{T} \times \mathbf{N}) \\ &= \kappa \left(\frac{ds}{dt} \right)^3 \mathbf{B}. & \mathbf{T} \times \mathbf{T} = \mathbf{0} \quad \text{and} \\ & & \mathbf{T} \times \mathbf{N} = \mathbf{B} \end{aligned}$$

It follows that

$$|\mathbf{v} \times \mathbf{a}| = \kappa \left| \frac{ds}{dt} \right|^3 |\mathbf{B}| = \kappa |\mathbf{v}|^3. \quad \frac{ds}{dt} = |\mathbf{v}| \quad \text{and} \quad |\mathbf{B}| = 1$$

Solving for κ gives the following formula.

Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \quad (5)$$

Equation (5) calculates the curvature, a geometric property of the curve, from the velocity and acceleration of any vector representation of the curve in which $|\mathbf{v}|$ is different from zero. Take a moment to think about how remarkable this really is: From any formula for motion along a curve, no matter how variable the motion may be (as long as \mathbf{v} is never zero), we can calculate a physical property of the curve that seems to have nothing to do with the way the curve is traversed.

The most widely used formula for torsion, derived in more advanced texts, is

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \quad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}). \quad (6)$$

Newton's Dot Notation for Derivatives

The dots in Equation (6) denote differentiation with respect to t , one derivative for each dot. Thus, \dot{x} ("x dot") means dx/dt , \ddot{x} ("x double dot") means d^2x/dt^2 , and \dddot{x} ("x triple dot") means d^3x/dt^3 . Similarly, $\dot{y} = dy/dt$, and so on.

This formula calculates the torsion directly from the derivatives of the component functions $x = f(t)$, $y = g(t)$, $z = h(t)$ that make up \mathbf{r} . The determinant's first row comes from \mathbf{v} , the second row comes from \mathbf{a} , and the third row comes from $\dot{\mathbf{a}} = d\mathbf{a}/dt$.

EXAMPLE 2 Finding Curvature and Torsion

Use Equations (5) and (6) to find κ and τ for the helix

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$

Solution We calculate the curvature with Equation (5):

$$\mathbf{v} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}$$

$$\mathbf{a} = -(a \cos t)\mathbf{i} - (a \sin t)\mathbf{j}$$

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \end{vmatrix} \\ &= (ab \sin t)\mathbf{i} - (ab \cos t)\mathbf{j} + a^2\mathbf{k} \\ \kappa &= \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{a^2b^2 + a^4}}{(a^2 + b^2)^{3/2}} = \frac{a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}. \end{aligned} \quad (7)$$

Notice that Equation (7) agrees with the result in Example 5 in Section 13.4, where we calculated the curvature directly from its definition.

To evaluate Equation (6) for the torsion, we find the entries in the determinant by differentiating \mathbf{r} with respect to t . We already have \mathbf{v} and \mathbf{a} , and

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (a \sin t)\mathbf{i} - (a \cos t)\mathbf{j}.$$

Hence,

$$\begin{aligned} \tau &= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{\begin{vmatrix} -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \\ a \sin t & -a \cos t & 0 \end{vmatrix}}{(a\sqrt{a^2 + b^2})^2} && \text{Value of } |\mathbf{v} \times \mathbf{a}| \\ &&& \text{from Equation (7)} \\ &= \frac{b(a^2 \cos^2 t + a^2 \sin^2 t)}{a^2(a^2 + b^2)} \\ &= \frac{b}{a^2 + b^2}. \end{aligned}$$

From this last equation we see that the torsion of a helix about a circular cylinder is constant. In fact, constant curvature and constant torsion characterize the helix among all curves in space. ■

Formulas for Curves in Space

Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature: $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Torsion: $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{\ddot{x}} & \ddot{\ddot{y}} & \ddot{\ddot{z}} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$

Tangential and normal scalar components of acceleration: $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$

$$a_T = \frac{d}{dt} |\mathbf{v}|$$

$$a_N = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$