EXERCISES 14.1

Domain, Range, and Level Curves

In Exercises 1–12, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine if the domain is an open region, a closed region, or neither, and **(f)** decide if the domain is bounded or unbounded.

1. $f(x, y) = y - x$

2. $f(x, y) = \sqrt{y - x}$

3. $f(x, y) = 4x^2 + 9y^2$

4. $f(x, y) = x^2 - y^2$

5. $f(x, y) = xy$

6. $f(x, y) = y/x^2$

7. $f(x, y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$

8. $f(x, y) = \sqrt{9 - x^2 - y^2}$

9. $f(x, y) = \ln(x^2 + y^2)$

10. $f(x, y) = e^{-(x^2+y^2)}$

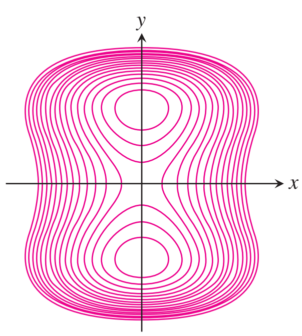
11. $f(x, y) = \sin^{-1}(y - x)$

12. $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$

Identifying Surfaces and Level Curves

Exercises 13–18 show level curves for the functions graphed in (a)–(f). Match each set of curves with the appropriate function.

13.



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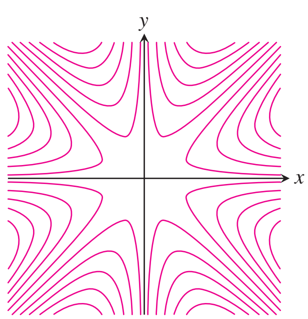

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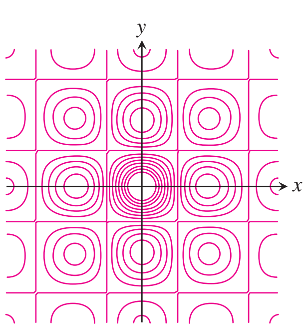

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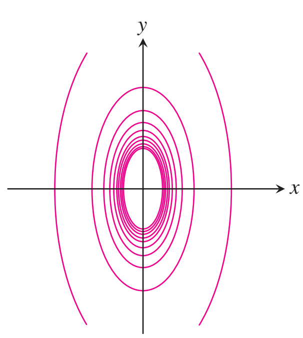

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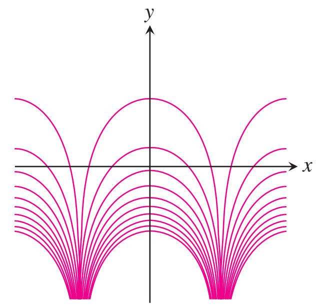

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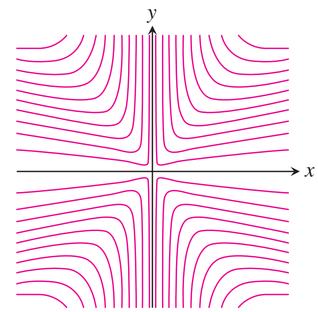
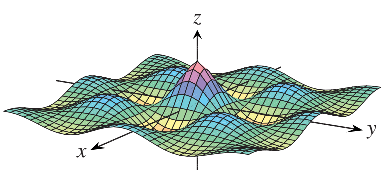
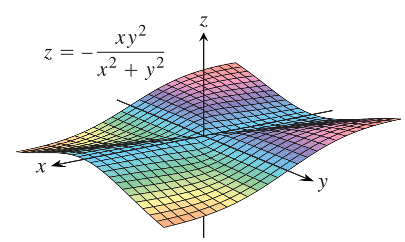


a.



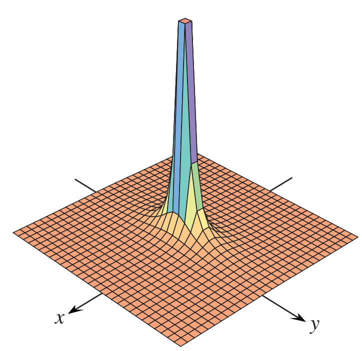
$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2 + y^2}/4}$

b.



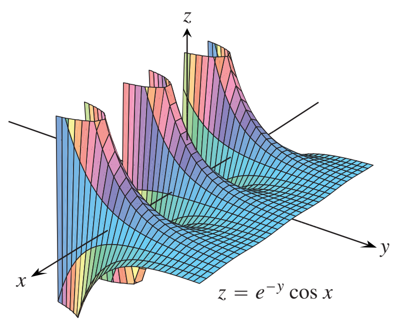
$z = -\dfrac{xy^2}{x^2 + y^2}$

c.

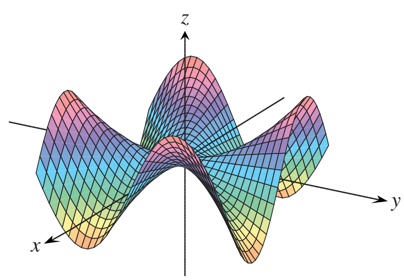


$z = \dfrac{1}{4x^2 + y^2}$

d.

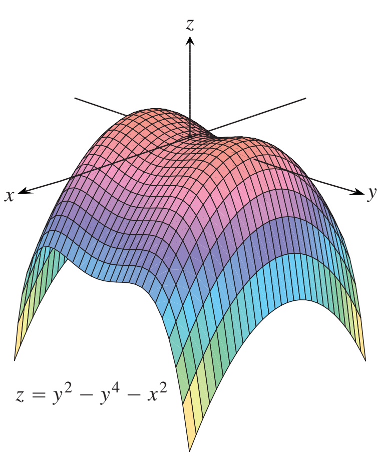


e.



$z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$

f.



Identifying Functions of Two Variables

Display the values of the functions in Exercises 19–28 in two ways: **(a)** by sketching the surface $z = f(x, y)$ and **(b)** by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.

19. $f(x, y) = y^2$ **20.** $f(x, y) = 4 - y^2$

21. $f(x, y) = x^2 + y^2$ **22.** $f(x, y) = \sqrt{x^2 + y^2}$

23. $f(x, y) = -(x^2 + y^2)$ **24.** $f(x, y) = 4 - x^2 - y^2$

25. $f(x, y) = 4x^2 + y^2$ **26.** $f(x, y) = 4x^2 + y^2 + 1$

27. $f(x, y) = 1 - |y|$ **28.** $f(x, y) = 1 - |x| - |y|$

Finding a Level Curve

In Exercises 29–32, find an equation for the level curve of the function $f(x, y)$ that passes through the given point.

29. $f(x, y) = 16 - x^2 - y^2, \quad (2\sqrt{2}, \sqrt{2})$

30. $f(x, y) = \sqrt{x^2 - 1}, \quad (1, 0)$

31. $f(x, y) = \displaystyle\int_x^y \frac{dt}{1 + t^2}, \quad (-\sqrt{2}, \sqrt{2})$

32. $f(x, y) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n, \quad (1, 2)$

Sketching Level Surfaces

In Exercises 33–40, sketch a typical level surface for the function.

33. $f(x, y, z) = x^2 + y^2 + z^2$ **34.** $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

35. $f(x, y, z) = x + z$ **36.** $f(x, y, z) = z$

37. $f(x, y, z) = x^2 + y^2$ **38.** $f(x, y, z) = y^2 + z^2$

39. $f(x, y, z) = z - x^2 - y^2$

40. $f(x, y, z) = (x^2/25) + (y^2/16) + (z^2/9)$

Finding a Level Surface

In Exercises 41–44, find an equation for the level surface of the function through the given point.

41. $f(x, y, z) = \sqrt{x - y} - \ln z, \quad (3, -1, 1)$

42. $f(x, y, z) = \ln(x^2 + y + z^2), \quad (-1, 2, 1)$

43. $g(x, y, z) = \displaystyle\sum_{n=0}^{\infty} \frac{(x + y)^n}{n!\, z^n}, \quad (\ln 2, \ln 4, 3)$

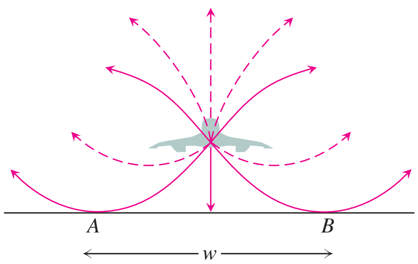
44. $g(x, y, z) = \displaystyle\int_x^y \frac{d\theta}{\sqrt{1 - \theta^2}} + \int_{\sqrt{2}}^z \frac{dt}{t\sqrt{t^2 - 1}}, \quad (0, 1/2, 2)$

Theory and Examples

45. The maximum value of a function on a line in space Does the function $f(x, y, z) = xyz$ have a maximum value on the line $x = 20 - t, y = t, z = 20$? If so, what is it? Give reasons for your answer. (*Hint:* Along the line, $w = f(x, y, z)$ is a differentiable function of t.)

46. The minimum value of a function on a line in space Does the function $f(x, y, z) = xy - z$ have a minimum value on the line $x = t - 1, y = t - 2, z = t + 7$? If so, what is it? Give reasons for your answer. (*Hint:* Along the line, $w = f(x, y, z)$ is a differentiable function of t.)

47. The Concorde's sonic booms Sound waves from the *Concorde* bend as the temperature changes above and below the altitude at which the plane flies. The sonic boom carpet is the region on the ground that receives shock waves directly from the plane, not reflected from the atmosphere or diffracted along the ground. The carpet is determined by the grazing rays striking the ground from the point directly under the plane. (See accompanying figure.)



Sonic boom carpet

The width w of the region in which people on the ground hear the *Concorde*'s sonic boom directly, not reflected from a layer in the atmosphere, is a function of

T = air temperature at ground level (in degrees Kelvin)

h = the *Concorde*'s altitude (in kilometers)

d = the vertical temperature gradient (temperature drop in degrees Kelvin per kilometer).

The formula for w is

$$w = 4\left(\frac{Th}{d}\right)^{1/2}.$$

The Washington-bound *Concorde* approached the United States from Europe on a course that took it south of Nantucket Island at an altitude of 16.8 km. If the surface temperature is 290 K and the vertical temperature gradient is 5 K/km, how many kilometers south of Nantucket did the plane have to be flown to keep its sonic boom carpet away from the island? (From "Concorde Sonic Booms as an Atmospheric Probe" by N. K. Balachandra, W. L. Donn, and D. H. Rind, *Science*, Vol. 197 (July 1, 1977), pp. 47–49.)

48. As you know, the graph of a real-valued function of a single real variable is a set in a two-coordinate space. The graph of a real-valued function of two independent real variables is a set in a three-coordinate space. The graph of a real-valued function of three independent real variables is a set in a four-coordinate space. How would you define the graph of a real-valued function $f(x_1, x_2, x_3, x_4)$ of four independent real variables? How would you define the graph of a real-valued function $f(x_1, x_2, x_3, \ldots, x_n)$ of n independent real variables?

COMPUTER EXPLORATIONS

Explicit Surfaces

Use a CAS to perform the following steps for each of the functions in Exercises 49–52.

a. Plot the surface over the given rectangle.

b. Plot several level curves in the rectangle.

c. Plot the level curve of f through the given point.

49. $f(x, y) = x \sin \dfrac{y}{2} + y \sin 2x, \quad 0 \le x \le 5\pi \quad 0 \le y \le 5\pi,$ $P(3\pi, 3\pi)$

50. $f(x, y) = (\sin x)(\cos y)e^{\sqrt{x^2+y^2}/8}, \quad 0 \le x \le 5\pi,$ $0 \le y \le 5\pi, \quad P(4\pi, 4\pi)$

51. $f(x, y) = \sin(x + 2\cos y), \quad -2\pi \le x \le 2\pi,$ $-2\pi \le y \le 2\pi, \quad P(\pi, \pi)$

52. $f(x, y) = e^{(x^{0.1} - y)} \sin(x^2 + y^2), \quad 0 \le x \le 2\pi,$ $-2\pi \le y \le \pi, \quad P(\pi, -\pi)$

Implicit Surfaces

Use a CAS to plot the level surfaces in Exercises 53–56.

53. $4 \ln(x^2 + y^2 + z^2) = 1$

54. $x^2 + z^2 = 1$

55. $x + y^2 - 3z^2 = 1$

56. $\sin\left(\dfrac{x}{2}\right) - (\cos y)\sqrt{x^2 + z^2} = 2$

Parametrized Surfaces

Just as you describe curves in the plane parametrically with a pair of equations $x = f(t), y = g(t)$ defined on some parameter interval I, you can sometimes describe surfaces in space with a triple of equations $x = f(u, v), y = g(u, v), z = h(u, v)$ defined on some parameter rectangle $a \le u \le b, c \le v \le d$. Many computer algebra systems permit you to plot such surfaces in *parametric mode*. (Parametrized surfaces are discussed in detail in Section 16.6.) Use a CAS to plot the surfaces in Exercises 57–60. Also plot several level curves in the xy-plane.

57. $x = u\cos v, \quad y = u \sin v, \quad z = u, \quad 0 \le u \le 2,$ $0 \le v \le 2\pi$

58. $x = u\cos v, \quad y = u \sin v, \quad z = v, \quad 0 \le u \le 2,$ $0 \le v \le 2\pi$

59. $x = (2 + \cos u)\cos v, \quad y = (2 + \cos u)\sin v, \quad z = \sin u,$ $0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$

60. $x = 2\cos u \cos v, \quad y = 2\cos u \sin v, \quad z = 2\sin u,$ $0 \le u \le 2\pi, \quad 0 \le v \le \pi$