14.2 Limits and Continuity in Higher Dimensions

This section treats limits and continuity for multivariable functions. The definition of the limit of a function of two or three variables is similar to the definition of the limit of a function of a single variable but with a crucial difference, as we now see.

Limits

If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0). This is similar to the informal definition for the limit of a function of a single variable. Notice, however, that if (x_0, y_0) lies in the interior of f's domain, (x, y) can approach (x_0, y_0) from any direction. The direction of approach can be an issue, as in some of the examples that follow.

DEFINITION Limit of a Function of Two Variables

We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

The definition of limit says that the distance between $f(x, y)$ and L becomes arbitrarily small whenever the distance from (x, y) to (x_0, y_0) is made sufficiently small (but not 0).

The definition of limit applies to boundary points (x_0, y_0) as well as interior points of the domain of f. The only requirement is that the point (x, y) remain in the domain at all times. It can be shown, as for functions of a single variable, that

$$\lim_{(x, y)\to(x_0, y_0)} x = x_0$$

$$\lim_{(x, y)\to(x_0, y_0)} y = y_0$$

$$\lim_{(x, y)\to(x_0, y_0)} k = k \qquad \text{(any number } k\text{).}$$

For example, in the first limit statement above, $f(x, y) = x$ and $L = x_0$. Using the definition of limit, suppose that $\epsilon > 0$ is chosen. If we let δ equal this ϵ, we see that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \epsilon$$

implies

$$0 < \sqrt{(x - x_0)^2} < \epsilon$$

$$|x - x_0| < \epsilon \qquad \sqrt{a^2} = |a|$$

$$|f(x, y) - x_0| < \epsilon \qquad x = f(x, y)$$

That is,

$$|f(x, y) - x_0| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = \lim_{(x, y) \to (x_0, y_0)} x = x_0.$$

It can also be shown that the limit of the sum of two functions is the sum of their limits (when they both exist), with similar results for the limits of the differences, products, constant multiples, quotients, and powers.

THEOREM 1 Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \qquad \text{and} \qquad \lim_{(x, y) \to (x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$
3. *Product Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
4. *Constant Multiple Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} (kf(x, y)) = kL$ (any number k)
5. *Quotient Rule:* $\displaystyle\lim_{(x, y) \to (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}$ $\quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, and $s \neq 0$, then

$$\lim_{(x, y) \to (x_0, y_0)} (f(x, y))^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

While we won't prove Theorem 1 here, we give an informal discussion of why it's true. If (x, y) is sufficiently close to (x_0, y_0), then $f(x, y)$ is close to L and $g(x, y)$ is close to M (from the informal interpretation of limits). It is then reasonable that $f(x, y) + g(x, y)$ is close to $L + M$; $f(x, y) - g(x, y)$ is close to $L - M$; $f(x, y)g(x, y)$ is close to LM; $kf(x, y)$ is close to kL; and that $f(x, y)/g(x, y)$ is close to L/M if $M \neq 0$.

When we apply Theorem 1 to polynomials and rational functions, we obtain the useful result that the limits of these functions as $(x, y) \to (x_0, y_0)$ can be calculated by evaluating the functions at (x_0, y_0). The only requirement is that the rational functions be defined at (x_0, y_0).

EXAMPLE 1 Calculating Limits

(a) $$\lim_{(x, y) \to (0,1)} \frac{x - xy + 3}{x^2 y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x, y) \to (3, -4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$ ■

EXAMPLE 2 Calculating Limits

Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{\left(x^2 - xy\right)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)\left(\sqrt{x} + \sqrt{y}\right)}{x - y} && \text{Algebra} \\
&= \lim_{(x,y)\to(0,0)} x\left(\sqrt{x} + \sqrt{y}\right) && \text{Cancel the nonzero factor } (x - y). \\
&= 0\left(\sqrt{0} + \sqrt{0}\right) = 0
\end{aligned}$$

We can cancel the factor $(x - y)$ because the path $y = x$ (along which $x - y = 0$) is *not* in the domain of the function

$$\frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

■

EXAMPLE 3 Applying the Limit Definition

Find $\lim_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2 + y^2}$ if it exists.

Solution We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0, 0)$, the value of the limit must be 0. To see if this is true, we apply the definition of limit.

Let $\epsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2 + y^2} < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2 + y^2} < \delta.$$

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2 + y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2}.$$

So if we choose $\delta = \epsilon/4$ and let $0 < \sqrt{x^2 + y^2} < \delta$, we get

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| \le 4\sqrt{x^2 + y^2} < 4\delta = 4\left(\frac{\epsilon}{4}\right) = \epsilon.$$

It follows from the definition that

$$\lim_{(x, y)\to(0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

Continuity

As with functions of a single variable, continuity is defined in terms of limits.

DEFINITION Continuous Function of Two Variables

A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

As with the definition of limit, the definition of continuity applies at boundary points as well as interior points of the domain of f. The only requirement is that the point (x, y) remain in the domain at all times.

As you may have guessed, one of the consequences of Theorem 1 is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, products, constant multiples, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined.

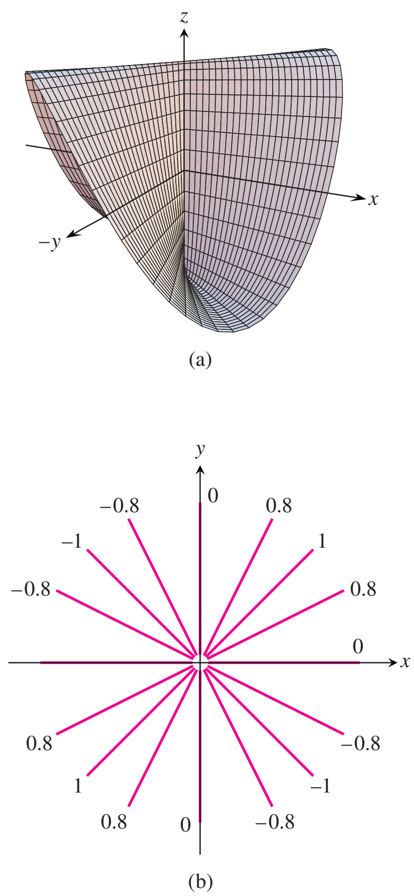

FIGURE 14.11 (a) The graph of

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

The function is continuous at every point except the origin. (b) The level curves of f (Example 4).

EXAMPLE 4 A Function with a Single Point of Discontinuity

You Try It

Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.11).

Solution The function f is continuous at any point $(x, y) \ne (0, 0)$ because its values are then given by a rational function of x and y.

At $(0, 0)$, the value of f is defined, but f, we claim, has no limit as $(x, y) \to (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.

For every value of m, the function f has a constant value on the "punctured" line $y = mx, x \neq 0$, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2 + y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Therefore, f has this number as its limit as (x, y) approaches $(0, 0)$ along the line:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

This limit changes with m. There is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous. ■

Example 4 illustrates an important point about limits of functions of two variables (or even more variables, for that matter). For a limit to exist at a point, the limit must be the same along every approach path. This result is analogous to the single-variable case where both the left- and right-sided limits had to have the same value; therefore, for functions of two or more variables, if we ever find paths with different limits, we know the function has no limit at the point they approach.

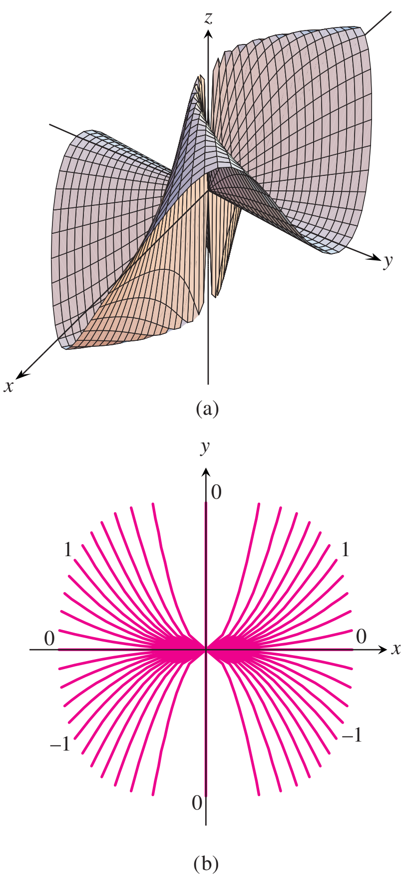

FIGURE 14.12 (a) The graph of $f(x, y) = 2x^2y/(x^4 + y^2)$. As the graph suggests and the level-curve values in part (b) confirm, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist (Example 5).

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

EXAMPLE 5 Applying the Two-Path Test

Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.12) has no limit as (x, y) approaches $(0, 0)$.

Solution The limit cannot be found by direct substitution, which gives the form $0/0$. We examine the values of f along curves that end at $(0, 0)$. Along the curve $y = kx^2, x \neq 0$, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches $(0, 0)$ along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches $(0, 0)$.

The language here may seem contradictory. You might well ask, "What do you mean f has no limit as (x, y) approaches the origin—it has lots of limits." But that is the point. There is no *single* path-independent limit, and therefore, by the definition, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. ■

Compositions of continuous functions are also continuous. The proof, omitted here, is similar to that for functions of a single variable (Theorem 10 in Section 2.6).

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

For example, the composite functions

$$e^{x-y}, \qquad \cos \frac{xy}{x^2 + 1}, \qquad \ln (1 + x^2 y^2)$$

are continuous at every point (x, y).

As with functions of a single variable, the general rule is that composites of continuous functions are continuous. The only requirement is that each function be continuous where it is applied.

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln (x + y + z) \qquad \text{and} \qquad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P\to(1,0,-1)} \frac{e^{x+z}}{z^2 + \cos \sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z), may be found by direct substitution.

Extreme Values of Continuous Functions on Closed, Bounded Sets

We have seen that a function of a single variable that is continuous throughout a closed, bounded interval $[a, b]$ takes on an absolute maximum value and an absolute minimum value at least once in $[a, b]$. The same is true of a function $z = f(x, y)$ that is continuous on a closed, bounded set R in the plane (like a line segment, a disk, or a filled-in triangle). The function takes on an absolute maximum value at some point in R and an absolute minimum value at some point in R.

Theorems similar to these and other theorems of this section hold for functions of three or more variables. A continuous function $w = f(x, y, z)$, for example, must take on absolute maximum and minimum values on any closed, bounded set (solid ball or cube, spherical shell, rectangular solid) on which it is defined.

We learn how to find these extreme values in Section 14.7, but first we need to study derivatives in higher dimensions. That is the topic of the next section.