

14.3

Partial Derivatives

The calculus of several variables is basically single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a “partial” derivative. This section shows how partial derivatives are defined and interpreted geometrically, and how to calculate them by applying the rules for differentiating functions of a single variable.

Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$ (Figure 14.13). This curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x ; the vertical coordinate is z . The y -value is held constant at y_0 , so y is not a variable.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$. To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the d previously used.

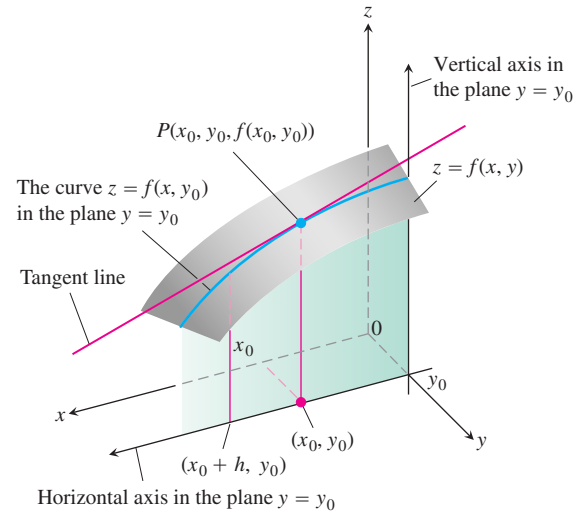


FIGURE 14.13 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy -plane.

DEFINITION Partial Derivative with Respect to x

The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

An equivalent expression for the partial derivative is

$$\left. \frac{d}{dx} f(x, y_0) \right|_{x=x_0}.$$

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0) . The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\partial f / \partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0 . This is the rate of change of f in the direction of \mathbf{i} at (x_0, y_0) .

The notation for a partial derivative depends on what we want to emphasize:

$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)}$ or $f_x(x_0, y_0)$ “Partial derivative of f with respect to x at (x_0, y_0) ” or “ f sub x at (x_0, y_0) .” Convenient for stressing the point (x_0, y_0) .

$\left. \frac{\partial z}{\partial x} \right|_{(x_0, y_0)}$ “Partial derivative of z with respect to x at (x_0, y_0) .” Common in science and engineering when you are dealing with variables and do not mention the function explicitly.

$f_x, \frac{\partial f}{\partial x}, z_x,$ or $\frac{\partial z}{\partial x}$ “Partial derivative of f (or z) with respect to x .” Convenient when you regard the partial derivative as a function in its own right.

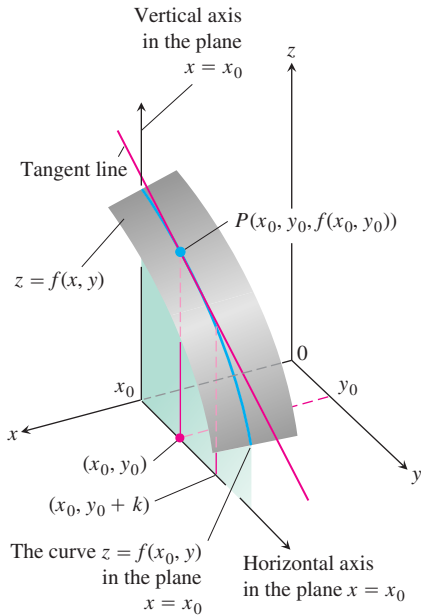


FIGURE 14.14 The intersection of the plane $x = x_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy -plane.

The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x . We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ with respect to y at y_0 .

DEFINITION Partial Derivative with Respect to y

The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = \left. \frac{d}{dy} f(x_0, y) \right|_{y=y_0} = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ (Figure 14.14) is the partial derivative of f with respect to y at (x_0, y_0) . The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative gives the rate of change of f with respect to y at (x_0, y_0) when x is held fixed at the value x_0 . This is the rate of change of f in the direction of \mathbf{j} at (x_0, y_0) .

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x :

$$\left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)}, \quad f_y(x_0, y_0), \quad \frac{\partial f}{\partial y}, \quad f_y.$$

Notice that we now have two tangent lines associated with the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ (Figure 14.15). Is the plane they determine tangent to the surface at P ? We will see that it is, but we have to learn more about partial derivatives before we can find out why.

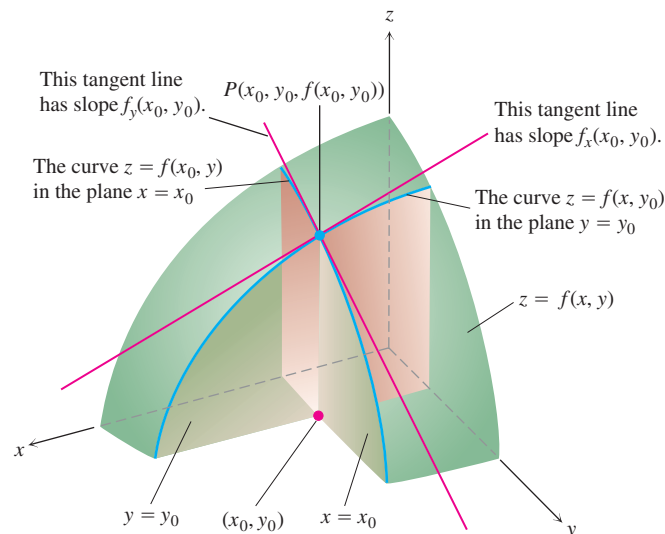


FIGURE 14.15 Figures 14.13 and 14.14 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

Calculations

The definitions of $\partial f/\partial x$ and $\partial f/\partial y$ give us two different ways of differentiating f at a point: with respect to x in the usual way while treating y as a constant and with respect to y in the usual way while treating x as constant. As the following examples show, the values of these partial derivatives are usually different at a given point (x_0, y_0) .

EXAMPLE 1 Finding Partial Derivatives at a Point

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x :

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y :

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. ■

EXAMPLE 2 Finding a Partial Derivative as a Function

Find $\partial f/\partial y$ if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y) \\ &= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy. \end{aligned} \quad \blacksquare$$

USING TECHNOLOGY Partial Differentiation

A simple grapher can support your calculations even in multiple dimensions. If you specify the values of all but one independent variable, the grapher can calculate partial derivatives and can plot traces with respect to that remaining variable. Typically, a CAS can compute partial derivatives symbolically and numerically as easily as it can compute simple derivatives. Most systems use the same command to differentiate a function, regardless of the number of variables. (Simply specify the variable with which differentiation is to take place).

EXAMPLE 3 Partial Derivatives May Be Different Functions

Find f_x and f_y if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we get

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial x} (2y) - 2y \frac{\partial}{\partial x} (y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}. \end{aligned}$$

With x held constant, we get

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial y} (2y) - 2y \frac{\partial}{\partial y} (y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}. \end{aligned}$$

Implicit differentiation works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

EXAMPLE 4 Implicit Partial Differentiation

Find $\partial z / \partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x , holding y constant and treating z as a differentiable function of x :

$$\frac{\partial}{\partial x} (yz) - \frac{\partial}{\partial x} \ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1 + 0$$

With y constant,
 $\frac{\partial}{\partial x} (yz) = y \frac{\partial z}{\partial x}$.

$$\left(y - \frac{1}{z} \right) \frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

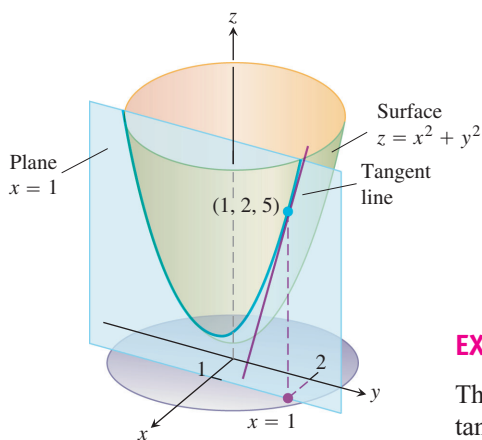


FIGURE 14.16 The tangent to the curve of intersection of the plane $x = 1$ and surface $z = x^2 + y^2$ at the point $(1, 2, 5)$ (Example 5).

EXAMPLE 5 Finding the Slope of a Surface in the y -Direction

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$ (Figure 14.16).

Solution The slope is the value of the partial derivative $\partial z / \partial y$ at $(1, 2)$:

$$\left. \frac{\partial z}{\partial y} \right|_{(1,2)} = \left. \frac{\partial}{\partial y} (x^2 + y^2) \right|_{(1,2)} = 2y \Big|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left. \frac{dz}{dy} \right|_{y=2} = \left. \frac{d}{dy} (1 + y^2) \right|_{y=2} = 2y \Big|_{y=2} = 4. \quad \blacksquare$$

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

EXAMPLE 6 A Function of Three Variables

If x , y , and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} [x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) \\ &= x \cos(y + 3z) \frac{\partial}{\partial z} (y + 3z) = 3x \cos(y + 3z). \end{aligned} \quad \blacksquare$$

EXAMPLE 7 Electrical Resistors in Parallel

If resistors of R_1 , R_2 , and R_3 ohms are connected in parallel to make an R -ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

(Figure 14.17). Find the value of $\partial R / \partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

Solution To find $\partial R / \partial R_2$, we treat R_1 and R_3 as constants and, using implicit differentiation, differentiate both sides of the equation with respect to R_2 :

$$\begin{aligned} \frac{\partial}{\partial R_2} \left(\frac{1}{R} \right) &= \frac{\partial}{\partial R_2} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \\ -\frac{1}{R^2} \frac{\partial R}{\partial R_2} &= 0 - \frac{1}{R_2^2} + 0 \\ \frac{\partial R}{\partial R_2} &= \frac{R^2}{R_2^2} = \left(\frac{R}{R_2} \right)^2. \end{aligned}$$

When $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$,

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3 + 2 + 1}{90} = \frac{6}{90} = \frac{1}{15},$$

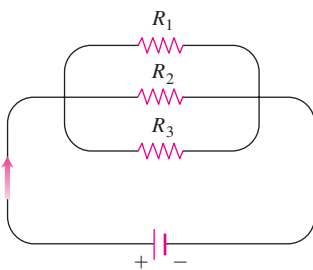


FIGURE 14.17 Resistors arranged this way are said to be connected in parallel (Example 7). Each resistor lets a portion of the current through. Their equivalent resistance R is calculated with the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

so $R = 15$ and

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}. \quad \blacksquare$$

Partial Derivatives and Continuity

A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without the function being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. If the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0) , however, then f is continuous at (x_0, y_0) , as we see at the end of this section.

EXAMPLE 8 Partials Exist, But f Discontinuous

Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.18).

- Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.
- Prove that f is not continuous at the origin.
- Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

Solution

- Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin), we have

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \Big|_{y=x} = \lim_{(x, y) \rightarrow (0, 0)} 0 = 0.$$

- Since $f(0, 0) = 1$, the limit in part (a) proves that f is not continuous at $(0, 0)$.
- To find $\partial f/\partial x$ at $(0, 0)$, we hold y fixed at $y = 0$. Then $f(x, y) = 1$ for all x , and the graph of f is the line L_1 in Figure 14.18. The slope of this line at any x is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at $(0, 0)$. Similarly, $\partial f/\partial y$ is the slope of line L_2 at any y , so $\partial f/\partial y = 0$ at $(0, 0)$. \blacksquare

Example 8 notwithstanding, it is still true in higher dimensions that *differentiability* at a point implies continuity. What Example 8 suggests is that we need a stronger requirement for differentiability in higher dimensions than the mere existence of the partial derivatives. We define differentiability for functions of two variables at the end of this section and revisit the connection to continuity.

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} & \quad \text{“}d \text{ squared } f dx \text{ squared”} & \quad \text{or} & \quad f_{xx} & \quad \text{“}f \text{ sub } xx\text{”} \\ \frac{\partial^2 f}{\partial y^2} & \quad \text{“}d \text{ squared } f dy \text{ squared”} & \quad \text{or} & \quad f_{yy} & \quad \text{“}f \text{ sub } yy\text{”} \end{aligned}$$

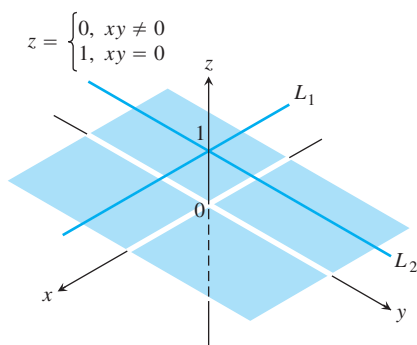


FIGURE 14.18 The graph of

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

consists of the lines L_1 and L_2 and the four open quadrants of the xy -plane. The function has partial derivatives at the origin but is not continuous there (Example 8).

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{“d squared } f dx dy\text{”} \quad \text{or} \quad f_{yx} \quad \text{“f sub } yx\text{”}$$

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{“d squared } f dy dx\text{”} \quad \text{or} \quad f_{xy} \quad \text{“f sub } xy\text{”}$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right),$$

and so on. Notice the order in which the derivatives are taken:

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{Differentiate first with respect to } y, \text{ then with respect to } x.$$

$$f_{yx} = (f_y)_x \quad \text{Means the same thing.}$$

HISTORICAL BIOGRAPHY

Pierre-Simon Laplace
(1749–1827)

EXAMPLE 9 Finding Second-Order Partial Derivatives

If $f(x, y) = x \cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (x \cos y + ye^x) \\ &= \cos y + ye^x \end{aligned}$$

So

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = -\sin y + e^x \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = ye^x. \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (x \cos y + ye^x) \\ &= -x \sin y + e^x \end{aligned}$$

So

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = -\sin y + e^x \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -x \cos y. \end{aligned} \quad \blacksquare$$

The Mixed Derivative Theorem

You may have noticed that the “mixed” second-order partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}$$

in Example 9 were equal. This was not a coincidence. They must be equal whenever f , f_x , f_y , f_{xy} , and f_{yx} are continuous, as stated in the following theorem.

THEOREM 2 The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x , f_y , f_{xy} , and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) , then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

HISTORICAL BIOGRAPHY

Alexis Clairaut
(1713–1765)

Theorem 2 is also known as Clairaut's Theorem, named after the French mathematician Alexis Clairaut who discovered it. A proof is given in Appendix 7. Theorem 2 says that to calculate a mixed second-order derivative, we may differentiate in either order, provided the continuity conditions are satisfied. This can work to our advantage.

EXAMPLE 10 Choosing the Order of Differentiation

Find $\partial^2 w / \partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w / \partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x . If we postpone the differentiation with respect to y and differentiate first with respect to x , however, we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to y , we obtain $\partial^2 w / \partial x \partial y = 1$ as well. ■

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

EXAMPLE 11 Calculating a Partial Derivative of Fourth-Order

Find f_{yyxz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y , then x , then y again, and finally with respect to z :

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$f_{yyxz} = -4$$

■

Differentiability

The starting point for differentiability is not Fermat's difference quotient but rather the idea of increment. You may recall from our work with functions of a single variable in Section 3.8 that if $y = f(x)$ is differentiable at $x = x_0$, then the change in the value of f that results from changing x from x_0 to $x_0 + \Delta x$ is given by an equation of the form

$$\Delta y = f'(x_0)\Delta x + \epsilon\Delta x$$

in which $\epsilon \rightarrow 0$ as $\Delta x \rightarrow 0$. For functions of two variables, the analogous property becomes the definition of differentiability. The Increment Theorem (from advanced calculus) tells us when to expect the property to hold.

THEOREM 3 The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0) . Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$.

You can see where the epsilons come from in the proof in Appendix 7. You will also see that similar results hold for functions of more than two independent variables.

DEFINITION Differentiable Function

A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$. We call f **differentiable** if it is differentiable at every point in its domain.

In light of this definition, we have the immediate corollary of Theorem 3 that a function is differentiable if its first partial derivatives are *continuous*.

COROLLARY OF THEOREM 3 Continuity of Partial Derivatives Implies Differentiability

If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R , then f is differentiable at every point of R .

If $z = f(x, y)$ is differentiable, then the definition of differentiability assures that $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ approaches 0 as Δx and Δy approach 0. This tells us that a function of two variables is continuous at every point where it is differentiable.

THEOREM 4 **Differentiability Implies Continuity**

If a function $f(x, y)$ is differentiable at (x_0, y_0) , then f is continuous at (x_0, y_0) .

As we can see from Theorems 3 and 4, a function $f(x, y)$ must be continuous at a point (x_0, y_0) if f_x and f_y are continuous throughout an open region containing (x_0, y_0) . Remember, however, that it is still possible for a function of two variables to be discontinuous at a point where its first partial derivatives exist, as we saw in Example 8. Existence alone of the partial derivative at a point is not enough.