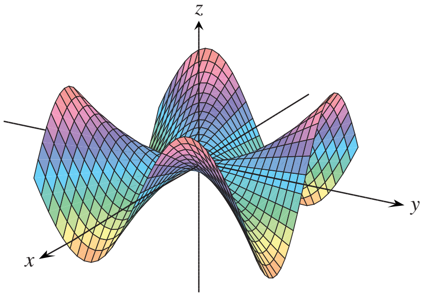
# Chapter 14 Additional and Advanced Exercises

## Partial Derivatives

1. **Function with saddle at the origin** If you did Exercise 50 in Section 14.2, you know that the function

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(see the accompanying figure) is continuous at (0, 0). Find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.



2. **Finding a function from second partials** Find a function $w = f(x, y)$ whose first partial derivatives are $\partial w/\partial x = 1 + e^x \cos y$ and $\partial w/\partial y = 2y - e^x \sin y$ and whose value at the point $(\ln 2, 0)$ is $\ln 2$.

3. **A proof of Leibniz's Rule** Leibniz's Rule says that if $f$ is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of $x$ whose values lie in $[a, b]$, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

Prove the rule by setting

$$g(u, v) = \int_u^v f(t)\,dt, \qquad u = u(x), \qquad v = v(x)$$

and calculating $dg/dx$ with the Chain Rule.

4. **Finding a function with constrained second partials** Suppose that $f$ is a twice-differentiable function of $r$, that $r = \sqrt{x^2 + y^2 + z^2}$, and that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

Show that for some constants $a$ and $b$,

$$f(r) = \frac{a}{r} + b.$$

5. **Homogeneous functions** A function $f(x, y)$ is *homogeneous of degree n* ($n$ a nonnegative integer) if $f(tx, ty) = t^n f(x, y)$ for all $t$, $x$, and $y$. For such a function (sufficiently differentiable), prove that

**a.** $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y)$

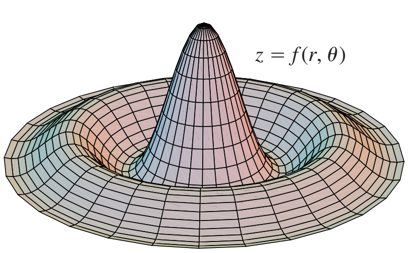
**b.** $x^2\left(\dfrac{\partial^2 f}{\partial x^2}\right) + 2xy\left(\dfrac{\partial^2 f}{\partial x \partial y}\right) + y^2\left(\dfrac{\partial^2 f}{\partial y^2}\right) = n(n-1)f.$

6. **Surface in polar coordinates** Let

$$f(r, \theta) = \begin{cases} \dfrac{\sin 6r}{6r}, & r \neq 0 \\ 1, & r = 0, \end{cases}$$

where $r$ and $\theta$ are polar coordinates. Find

**a.** $\lim_{r \to 0} f(r, \theta)$ **b.** $f_r(0, 0)$ **c.** $f_\theta(r, \theta), \quad r \neq 0.$




## Gradients and Tangents

7. **Properties of position vectors** Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $r = |\mathbf{r}|$.

**a.** Show that $\nabla r = \mathbf{r}/r$.

**b.** Show that $\nabla(r^n) = nr^{n-2}\mathbf{r}$.

**c.** Find a function whose gradient equals $\mathbf{r}$.

**d.** Show that $\mathbf{r} \cdot d\mathbf{r} = r\,dr$.

**e.** Show that $\nabla(\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}$ for any constant vector $\mathbf{A}$.

8. **Gradient orthogonal to tangent** Suppose that a differentiable function $f(x, y)$ has the constant value $c$ along the differentiable curve $x = g(t), y = h(t)$; that is

$$f(g(t), h(t)) = c$$

for all values of $t$. Differentiate both sides of this equation with respect to $t$ to show that $\nabla f$ is orthogonal to the curve's tangent vector at every point on the curve.

9. **Curve tangent to a surface** Show that the curve

$$\mathbf{r}(t) = (\ln t)\mathbf{i} + (t \ln t)\mathbf{j} + t\mathbf{k}$$

is tangent to the surface

$$xz^2 - yz + \cos xy = 1$$

at $(0, 0, 1)$.

10. **Curve tangent to a surface** Show that the curve

$$\mathbf{r}(t) = \left(\frac{t^3}{4} - 2\right)\mathbf{i} + \left(\frac{4}{t} - 3\right)\mathbf{j} + \cos(t - 2)\mathbf{k}$$

is tangent to the surface

$$x^3 + y^3 + z^3 - xyz = 0$$

at $(0, -1, 1)$.

## Extreme Values

11. **Extrema on a surface** Show that the only possible maxima and minima of $z$ on the surface $z = x^3 + y^3 - 9xy + 27$ occur at $(0, 0)$ and $(3, 3)$. Show that neither a maximum nor a minimum occurs at $(0, 0)$. Determine whether $z$ has a maximum or a minimum at $(3, 3)$.

12. **Maximum in closed first quadrant** Find the maximum value of $f(x, y) = 6xye^{-(2x+3y)}$ in the closed first quadrant (includes the nonnegative axes).

13. **Minimum volume cut from first octant** Find the minimum volume for a region bounded by the planes $x = 0, y = 0, z = 0$ and a plane tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at a point in the first octant.

**14. Minimum distance from line to parabola in *xy*-plane** By minimizing the function $f(x, y, u, v) = (x - u)^2 + (y - v)^2$ subject to the constraints $y = x + 1$ and $u = v^2$, find the minimum distance in the $xy$-plane from the line $y = x + 1$ to the parabola $y^2 = x$.

## Theory and Examples

**15. Boundedness of first partials implies continuity** Prove the following theorem: If $f(x, y)$ is defined in an open region $R$ of the $xy$-plane and if $f_x$ and $f_y$ are bounded on $R$, then $f(x, y)$ is continuous on $R$. (The assumption of boundedness is essential.)

**16.** Suppose that $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve in the domain of a differentiable function $f(x, y, z)$. Describe the relation between $df/dt$, $\nabla f$, and $\mathbf{v} = d\mathbf{r}/dt$. What can be said about $\nabla f$ and $\mathbf{v}$ at interior points of the curve where $f$ has extreme values relative to its other values on the curve? Give reasons for your answer.

**17. Finding functions from partial derivatives** Suppose that $f$ and $g$ are functions of $x$ and $y$ such that

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y},$$

and suppose that

$$\frac{\partial f}{\partial x} = 0, \quad f(1, 2) = g(1, 2) = 5 \quad \text{and} \quad f(0, 0) = 4.$$

Find $f(x, y)$ and $g(x, y)$.

**18. Rate of change of the rate of change** We know that if $f(x, y)$ is a function of two variables and if $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$ is the rate of change of $f(x, y)$ at $(x, y)$ in the direction of $\mathbf{u}$. Give a similar formula for the rate of change *of the rate of change of* $f(x, y)$ at $(x, y)$ in the direction $\mathbf{u}$.

**19. Path of a heat-seeking particle** A heat-seeking particle has the property that at any point $(x, y)$ in the plane it moves in the direction of maximum temperature increase. If the temperature at $(x, y)$ is $T(x, y) = -e^{-2y} \cos x$, find an equation $y = f(x)$ for the path of a heat-seeking particle at the point $(\pi/4, 0)$.

**20. Velocity after a ricochet** A particle traveling in a straight line with constant velocity $\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ passes through the point $(0, 0, 30)$ and hits the surface $z = 2x^2 + 3y^2$. The particle ricochets off the surface, the angle of reflection being equal to the angle of incidence. Assuming no loss of speed, what is the velocity of the particle after the ricochet? Simplify your answer.

**21. Directional derivatives tangent to a surface** Let $S$ be the surface that is the graph of $f(x, y) = 10 - x^2 - y^2$. Suppose that the temperature in space at each point $(x, y, z)$ is $T(x, y, z) = x^2 y + y^2 z + 4x + 14y + z$.

**a.** Among all the possible directions tangential to the surface $S$ at the point $(0, 0, 10)$, which direction will make the rate of change of temperature at $(0, 0, 10)$ a maximum?

**b.** Which direction tangential to $S$ at the point $(1, 1, 8)$ will make the rate of change of temperature a maximum?

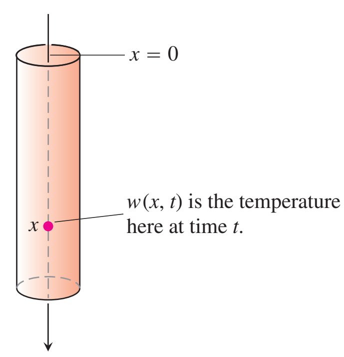
**22. Drilling another borehole** On a flat surface of land, geologists drilled a borehole straight down and hit a mineral deposit at 1000 ft. They drilled a second borehole 100 ft to the north of the first and hit the mineral deposit at 950 ft. A third borehole 100 ft east of the first borehole struck the mineral deposit at 1025 ft. The geologists have reasons to believe that the mineral deposit is in the shape of a dome, and for the sake of economy, they would like to find where the deposit is closest to the surface. Assuming the surface to be the $xy$-plane, in what direction from the first borehole would you suggest the geologists drill their fourth borehole?

## The One-Dimensional Heat Equation

If $w(x, t)$ represents the temperature at position $x$ at time $t$ in a uniform conducting rod with perfectly insulated sides (see the accompanying figure), then the partial derivatives $w_{xx}$ and $w_t$ satisfy a differential equation of the form

$$w_{xx} = \frac{1}{c^2} w_t.$$

This equation is called the **one-dimensional heat equation**. The value of the positive constant $c^2$ is determined by the material from which the rod is made. It has been determined experimentally for a broad range of materials. For a given application, one finds the appropriate value in a table. For dry soil, for example, $c^2 = 0.19 \text{ ft}^2/\text{day}$.




In chemistry and biochemistry, the heat equation is known as the **diffusion equation**. In this context, $w(x, t)$ represents the concentration of a dissolved substance, a salt for instance, diffusing along a tube filled with liquid. The value of $w(x, t)$ is the concentration at point $x$ at time $t$. In other applications, $w(x, t)$ represents the diffusion of a gas down a long, thin pipe.

In electrical engineering, the heat equation appears in the forms

$$v_{xx} = RCv_t$$

and

$$i_{xx} = RCi_t.$$

These equations describe the voltage $v$ and the flow of current $i$ in a coaxial cable or in any other cable in which leakage and inductance are negligible. The functions and constants in these equations are

$$
\begin{aligned}
v(x, t) &= \text{voltage at point } x \text{ at time } t \\
R &= \text{resistance per unit length} \\
C &= \text{capacitance to ground per unit of cable length} \\
i(x, t) &= \text{current at point } x \text{ at time } t.
\end{aligned}
$$

**23.** Find all solutions of the one-dimensional heat equation of the form $w = e^{rt} \sin \pi x$, where $r$ is a constant.

**24.** Find all solutions of the one-dimensional heat equation that have the form $w = e^{rt} \sin kx$ and satisfy the conditions that $w(0, t) = 0$ and $w(L, t) = 0$. What happens to these solutions as $t \to \infty$?