EXERCISES 15.1

Finding Regions of Integration and Double Integrals

In Exercises 1–10, sketch the region of integration and evaluate the integral.

1. $\displaystyle\int_0^3\int_0^2 (4 - y^2)\,dy\,dx$

2. $\displaystyle\int_0^3\int_{-2}^0 (x^2y - 2xy)\,dy\,dx$

3. $\displaystyle\int_{-1}^0\int_{-1}^1 (x + y + 1)\,dx\,dy$

4. $\displaystyle\int_{\pi}^{2\pi}\int_0^{\pi} (\sin x + \cos y)\,dx\,dy$

5. $\displaystyle\int_0^{\pi}\int_0^{x} x\sin y\,dy\,dx$

6. $\displaystyle\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx$

7. $\displaystyle\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$

8. $\displaystyle\int_1^2\int_y^{y^2} dx\,dy$

9. $\displaystyle\int_0^1\int_0^{y^2} 3y^3e^{xy}\,dx\,dy$

10. $\displaystyle\int_1^4\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dy\,dx$

In Exercises 11–16, integrate f over the given region.

11. Quadrilateral $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x, y = 2x, x = 1, x = 2$

12. Square $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2$, $1 \le y \le 2$

13. Triangle $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$

14. Rectangle $f(x, y) = y\cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

15. Triangle $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$

16. Curved region $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

Each of Exercises 17–20 gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

17. $\displaystyle\int_{-2}^0\int_v^{-v} 2\,dp\,dv$ (the pv-plane)

18. $\displaystyle\int_0^1\int_0^{\sqrt{1-s^2}} 8t\,dt\,ds$ (the st-plane)

19. $\displaystyle\int_{-\pi/3}^{\pi/3}\int_0^{\sec t} 3\cos t\,du\,dt$ (the tu-plane)

20. $\displaystyle\int_0^3\int_1^{4-2u} \frac{4 - 2u}{v^2}\,dv\,du$ (the uv-plane)

Reversing the Order of Integration

In Exercises 21–30, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

21. $\displaystyle\int_0^1\int_2^{4-2x} dy\,dx$

22. $\displaystyle\int_0^2\int_{y-2}^0 dx\,dy$

23. $\displaystyle\int_0^1\int_y^{\sqrt{y}} dx\,dy$

24. $\displaystyle\int_0^1\int_{1-x}^{1-x^2} dy\,dx$

25. $\displaystyle\int_0^1\int_1^{e^x} dy\,dx$

26. $\displaystyle\int_0^{\ln 2}\int_{e^y}^2 dx\,dy$

27. $\displaystyle\int_0^{3/2}\int_0^{9-4x^2} 16x\,dy\,dx$

28. $\displaystyle\int_0^2\int_0^{4-y^2} y\,dx\,dy$

29. $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\,dx\,dy$

30. $\displaystyle\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\,dy\,dx$

Evaluating Double Integrals

In Exercises 31–40, sketch the region of integration, reverse the order of integration, and evaluate the integral.

31. $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$

32. $\displaystyle\int_0^2\int_x^2 2y^2\sin xy\,dy\,dx$

33. $\displaystyle\int_0^1\int_y^1 x^2e^{xy}\,dx\,dy$

34. $\displaystyle\int_0^2\int_0^{4-x^2} \frac{xe^{2y}}{4 - y}\,dy\,dx$

35. $\displaystyle\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\,dx\,dy$

36. $\displaystyle\int_0^3\int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx$

37. $\displaystyle\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,dx\,dy$

38. $\displaystyle\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4 + 1}$

39. Square region $\iint_R (y - 2x^2)\,dA$ where R is the region bounded by the square $|x| + |y| = 1$

40. Triangular region $\iint_R xy\,dA$ where R is the region bounded by the lines $y = x$, $y = 2x$, and $x + y = 2$

Volume Beneath a Surface $z = f(x, y)$

41. Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

42. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

43. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

44. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

45. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

46. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

47. Find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

48. Find the volume of the solid cut from the square column $|x| + |y| \leq 1$ by the planes $z = 0$ and $3x + z = 3$.

49. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$ and $x = 1$, on the sides by the cylinders $y = \pm 1/x$, and above and below by the planes $z = x + 1$ and $z = 0$.

50. Find the volume of the solid bounded on the front and back by the planes $x = \pm\pi/3$, on the sides by the cylinders $y = \pm\sec x$, above by the cylinder $z = 1 + y^2$, and below by the xy-plane.

Integrals over Unbounded Regions

Improper double integrals can often be computed similarly to improper integrals of one variable. The first iteration of the following improper integrals is conducted just as if they were proper integrals. One then evaluates an improper integral of a single variable by taking appropriate limits, as in Section 8.8. Evaluate the improper integrals in Exercises 51–54 as iterated integrals.

51. $\displaystyle\int_1^{\infty}\int_{e^{-x}}^{1} \frac{1}{x^3 y}\, dy\, dx$

52. $\displaystyle\int_{-1}^{1}\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y + 1)\, dy\, dx$

53. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(y^2 + 1)}\, dx\, dy$

54. $\displaystyle\int_0^{\infty}\int_0^{\infty} xe^{-(x+2y)}\, dx\, dy$

Approximating Double Integrals

In Exercises 55 and 56, approximate the double integral of $f(x, y)$ over the region R partitioned by the given vertical lines $x = a$ and horizontal lines $y = c$. In each subrectangle, use (x_k, y_k) as indicated for your approximation.

$$\iint_R f(x, y)\, dA \approx \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

55. $f(x, y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1 - x^2}$ and below by the x-axis, using the partition $x = -1, -1/2, 0, 1/4, 1/2, 1$ and $y = 0, 1/2, 1$ with (x_k, y_k) the lower left corner in the kth subrectangle (provided the subrectangle lies within R)

56. $f(x, y) = x + 2y$ over the region R inside the circle $(x - 2)^2 + (y - 3)^2 = 1$ using the partition $x = 1, 3/2, 2, 5/2, 3$ and $y = 2, 5/2, 3, 7/2, 4$ with (x_k, y_k) the center (centroid) in the kth subrectangle (provided the subrectangle lies within R)

Theory and Examples

57. Circular sector Integrate $f(x, y) = \sqrt{4 - x^2}$ over the smaller sector cut from the disk $x^2 + y^2 \leq 4$ by the rays $\theta = \pi/6$ and $\theta = \pi/2$.

58. Unbounded region Integrate $f(x, y) = 1/[(x^2 - x)(y - 1)^{2/3}]$ over the infinite rectangle $2 \leq x < \infty, 0 \leq y \leq 2$.

59. Noncircular cylinder A solid right (noncircular) cylinder has its base R in the xy-plane and is bounded above by the paraboloid $z = x^2 + y^2$. The cylinder's volume is

$$V = \int_0^1\int_0^y (x^2 + y^2)\, dx\, dy + \int_1^2\int_0^{2-y} (x^2 + y^2)\, dx\, dy.$$

Sketch the base region R and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.

60. Converting to a double integral Evaluate the integral

$$\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\, dx.$$

(*Hint:* Write the integrand as an integral.)

61. Maximizing a double integral What region R in the xy-plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2)\, dA?$$

Give reasons for your answer.

62. Minimizing a double integral What region R in the xy-plane minimizes the value of

$$\iint_R (x^2 + y^2 - 9)\, dA?$$

Give reasons for your answer.

63. Is it possible to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the xy-plane and get different answers depending on the order of integration? Give reasons for your answer.

64. How would you evaluate the double integral of a continuous function $f(x, y)$ over the region R in the xy-plane enclosed by the triangle with vertices $(0, 1)$, $(2, 0)$, and $(1, 2)$? Give reasons for your answer.

65. Unbounded region Prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \lim_{b\to\infty}\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\, dx\, dy$$

$$= 4\left(\int_0^{\infty} e^{-x^2}\, dx\right)^2.$$

66. Improper double integral Evaluate the improper integral

$$\int_0^1\int_0^3 \frac{x^2}{(y - 1)^{2/3}}\, dy\, dx.$$

COMPUTER EXPLORATIONS

Evaluating Double Integrals Numerically

Use a CAS double-integral evaluator to estimate the values of the integrals in Exercises 67–70.

67. $\int_1^3 \int_1^x \frac{1}{xy}\,dy\,dx$

68. $\int_0^1 \int_0^1 e^{-(x^2+y^2)}\,dy\,dx$

69. $\int_0^1 \int_0^1 \tan^{-1} xy\,dy\,dx$

70. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1 - x^2 - y^2}\,dy\,dx$

Use a CAS double-integral evaluator to find the integrals in Exercises 71–76. Then reverse the order of integration and evaluate, again with a CAS.

71. $\int_0^1 \int_{2y}^4 e^{x^2}\,dx\,dy$

72. $\int_0^3 \int_{x^2}^9 x\cos(y^2)\,dy\,dx$

73. $\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\,dx\,dy$

74. $\int_0^2 \int_0^{4-y^2} e^{xy}\,dx\,dy$

75. $\int_1^2 \int_0^{x^2} \frac{1}{x+y}\,dy\,dx$

76. $\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$