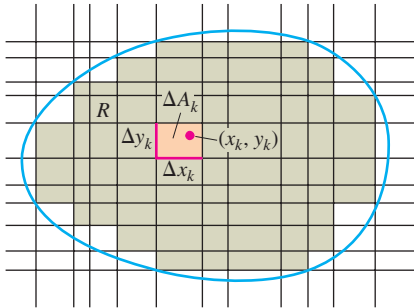


## 15.2 Area, Moments, and Centers of Mass



**FIGURE 15.14** As the norm of a partition of the region  $R$  approaches zero, the sum of the areas  $\Delta A_k$  gives the area of  $R$  defined by the double integral  $\iint_R dA$ .

In this section, we show how to use double integrals to calculate the areas of bounded regions in the plane and to find the average value of a function of two variables. Then we study the physical problem of finding the center of mass of a thin plate covering a region in the plane.

### Areas of Bounded Regions in the Plane

If we take  $f(x, y) = 1$  in the definition of the double integral over a region  $R$  in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k = \sum_{k=1}^n \Delta A_k. \quad (1)$$

This is simply the sum of the areas of the small rectangles in the partition of  $R$ , and approximates what we would like to call the area of  $R$ . As the norm of a partition of  $R$  approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of  $R$  becomes increasingly complete (Figure 15.14). We define the area of  $R$  to be the limit

$$\text{Area} = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \Delta A_k = \iint_R dA \quad (2)$$

#### DEFINITION Area

The **area** of a closed, bounded plane region  $R$  is

$$A = \iint_R dA.$$

As with the other definitions in this chapter, the definition here applies to a greater variety of regions than does the earlier single-variable definition of area, but it agrees with the earlier definition on regions to which they both apply. To evaluate the integral in the definition of area, we integrate the constant function  $f(x, y) = 1$  over  $R$ .

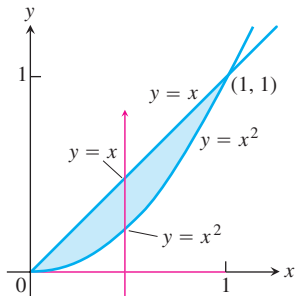


FIGURE 15.15 The region in Example 1.

**EXAMPLE 1** Finding Area

Find the area of the region  $R$  bounded by  $y = x$  and  $y = x^2$  in the first quadrant.

**Solution** We sketch the region (Figure 15.15), noting where the two curves intersect, and calculate the area as

$$\begin{aligned} A &= \int_0^1 \int_{x^2}^x dy \, dx = \int_0^1 \left[ y \right]_{x^2}^x dx \\ &= \int_0^1 (x - x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}. \end{aligned}$$

Notice that the single integral  $\int_0^1 (x - x^2) dx$ , obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.5. ■

**EXAMPLE 2** Finding Area

Find the area of the region  $R$  enclosed by the parabola  $y = x^2$  and the line  $y = x + 2$ .

**Solution** If we divide  $R$  into the regions  $R_1$  and  $R_2$  shown in Figure 15.16a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-x^2}^{\sqrt{y}} dx \, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx \, dy.$$

On the other hand, reversing the order of integration (Figure 15.16b) gives

$$A = \int_{-1}^2 \int_{x^2}^{x+2} dy \, dx.$$

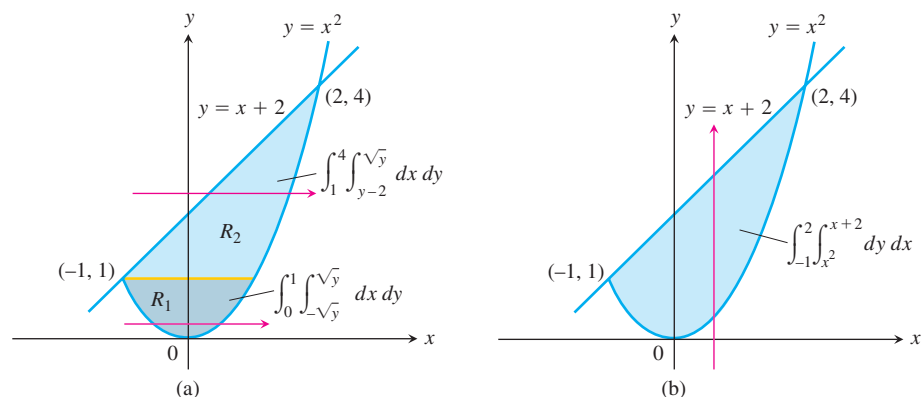


FIGURE 15.16 Calculating this area takes (a) two double integrals if the first integration is with respect to  $x$ , but (b) only one if the first integration is with respect to  $y$  (Example 2).

This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

$$A = \int_{-1}^2 \left[ y \right]_{x^2}^{x+2} dx = \int_{-1}^2 (x + 2 - x^2) dx = \left[ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}. \quad \blacksquare$$

### Average Value

The average value of an integrable function of one variable on a closed interval is the integral of the function over the interval divided by the length of the interval. For an integrable function of two variables defined on a bounded region in the plane, the average value is the integral over the region divided by the area of the region. This can be visualized by thinking of the function as giving the height at one instant of some water sloshing around in a tank whose vertical walls lie over the boundary of the region. The average height of the water in the tank can be found by letting the water settle down to a constant height. The height is then equal to the volume of water in the tank divided by the area of  $R$ . We are led to define the average value of an integrable function  $f$  over a region  $R$  to be

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA. \quad (3)$$

If  $f$  is the temperature of a thin plate covering  $R$ , then the double integral of  $f$  over  $R$  divided by the area of  $R$  is the plate's average temperature. If  $f(x, y)$  is the distance from the point  $(x, y)$  to a fixed point  $P$ , then the average value of  $f$  over  $R$  is the average distance of points in  $R$  from  $P$ .

#### EXAMPLE 3 Finding Average Value

Find the average value of  $f(x, y) = x \cos xy$  over the rectangle  $R: 0 \leq x \leq \pi, 0 \leq y \leq 1$ .

**Solution** The value of the integral of  $f$  over  $R$  is

$$\begin{aligned} \int_0^\pi \int_0^1 x \cos xy \, dy \, dx &= \int_0^\pi \left[ \sin xy \right]_{y=0}^{y=1} dx && \int x \cos xy \, dy = \sin xy + C \\ &= \int_0^\pi (\sin x - 0) \, dx = -\cos x \Big|_0^\pi = 1 + 1 = 2. \end{aligned}$$

The area of  $R$  is  $\pi$ . The average value of  $f$  over  $R$  is  $2/\pi$ . \blacksquare

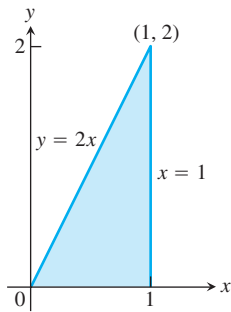
### Moments and Centers of Mass for Thin Flat Plates

In Section 6.4 we introduced the concepts of moments and centers of mass, and we saw how to compute these quantities for thin rods or strips and for plates of constant density. Using multiple integrals we can extend these calculations to a great variety of shapes with varying density. We first consider the problem of finding the center of mass of a thin flat plate: a disk of aluminum, say, or a triangular sheet of metal. We assume the distribution of

mass in such a plate to be continuous. A material's *density* function, denoted by  $\delta(x, y)$ , is the mass per unit area. The *mass* of a plate is obtained by integrating the density function over the region  $R$  forming the plate. The first moment about an axis is calculated by integrating over  $R$  the distance from the axis times the density. The center of mass is found from the first moments. Table 15.1 gives the double integral formulas for mass, first moments, and center of mass.

**TABLE 15.1** Mass and first moment formulas for thin plates covering a region  $R$  in the  $xy$ -plane

<b>Mass:</b>	$M = \iint_R \delta(x, y) dA$	$\delta(x, y)$ is the density at $(x, y)$
<b>First moments:</b>	$M_x = \iint_R y\delta(x, y) dA,$	$M_y = \iint_R x\delta(x, y) dA$
<b>Center of mass:</b>	$\bar{x} = \frac{M_y}{M},$	$\bar{y} = \frac{M_x}{M}$



**FIGURE 15.17** The triangular region covered by the plate in Example 4.

#### EXAMPLE 4 Finding the Center of Mass of a Thin Plate of Variable Density

A thin plate covers the triangular region bounded by the  $x$ -axis and the lines  $x = 1$  and  $y = 2x$  in the first quadrant. The plate's density at the point  $(x, y)$  is  $\delta(x, y) = 6x + 6y + 6$ . Find the plate's mass, first moments, and center of mass about the coordinate axes.

**Solution** We sketch the plate and put in enough detail to determine the limits of integration for the integrals we have to evaluate (Figure 15.17).

The plate's mass is

$$\begin{aligned} M &= \int_0^1 \int_0^{2x} \delta(x, y) dy dx = \int_0^1 \int_0^{2x} (6x + 6y + 6) dy dx \\ &= \int_0^1 \left[ 6xy + 3y^2 + 6y \right]_{y=0}^{y=2x} dx \\ &= \int_0^1 (24x^2 + 12x) dx = \left[ 8x^3 + 6x^2 \right]_0^1 = 14. \end{aligned}$$

The first moment about the  $x$ -axis is

$$\begin{aligned} M_x &= \int_0^1 \int_0^{2x} y\delta(x, y) dy dx = \int_0^1 \int_0^{2x} (6xy + 6y^2 + 6y) dy dx \\ &= \int_0^1 \left[ 3xy^2 + 2y^3 + 3y^2 \right]_{y=0}^{y=2x} dx = \int_0^1 (28x^3 + 12x^2) dx \\ &= \left[ 7x^4 + 4x^3 \right]_0^1 = 11. \end{aligned}$$

A similar calculation gives the moment about the  $y$ -axis:

$$M_y = \int_0^1 \int_0^{2x} x \delta(x, y) dy dx = 10.$$

The coordinates of the center of mass are therefore

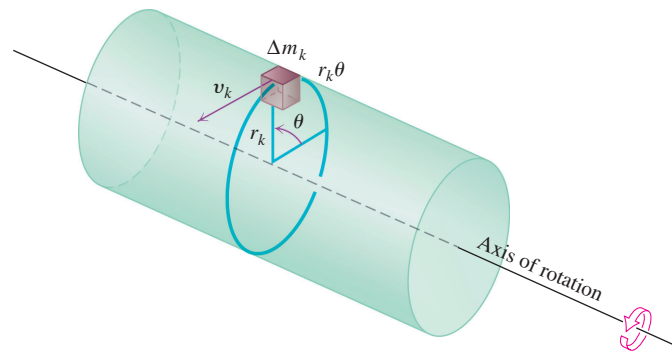
$$\bar{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7}, \quad \bar{y} = \frac{M_x}{M} = \frac{11}{14}. \quad \blacksquare$$

### Moments of Inertia

A body's first moments (Table 15.1) tell us about balance and about the torque the body exerts about different axes in a gravitational field. If the body is a rotating shaft, however, we are more likely to be interested in how much energy is stored in the shaft or about how much energy it will take to accelerate the shaft to a particular angular velocity. This is where the second moment or moment of inertia comes in.

Think of partitioning the shaft into small blocks of mass  $\Delta m_k$  and let  $r_k$  denote the distance from the  $k$ th block's center of mass to the axis of rotation (Figure 15.18). If the shaft rotates at an angular velocity of  $\omega = d\theta/dt$  radians per second, the block's center of mass will trace its orbit at a linear speed of

$$v_k = \frac{d}{dt}(r_k \theta) = r_k \frac{d\theta}{dt} = r_k \omega.$$



**FIGURE 15.18** To find an integral for the amount of energy stored in a rotating shaft, we first imagine the shaft to be partitioned into small blocks. Each block has its own kinetic energy. We add the contributions of the individual blocks to find the kinetic energy of the shaft.

The block's kinetic energy will be approximately

$$\frac{1}{2} \Delta m_k v_k^2 = \frac{1}{2} \Delta m_k (r_k \omega)^2 = \frac{1}{2} \omega^2 r_k^2 \Delta m_k.$$

The kinetic energy of the shaft will be approximately

$$\sum \frac{1}{2} \omega^2 r_k^2 \Delta m_k.$$

The integral approached by these sums as the shaft is partitioned into smaller and smaller blocks gives the shaft's kinetic energy:

$$KE_{\text{shaft}} = \int \frac{1}{2} \omega^2 r^2 dm = \frac{1}{2} \omega^2 \int r^2 dm. \quad (4)$$

The factor

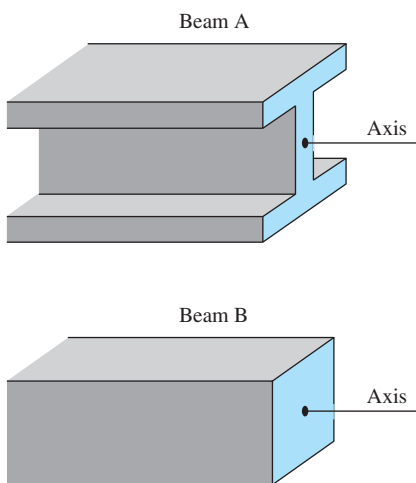
$$I = \int r^2 dm$$

is the *moment of inertia* of the shaft about its axis of rotation, and we see from Equation (4) that the shaft's kinetic energy is

$$KE_{\text{shaft}} = \frac{1}{2} I \omega^2.$$

The moment of inertia of a shaft resembles in some ways the inertia of a locomotive. To start a locomotive with mass  $m$  moving at a linear velocity  $v$ , we need to provide a kinetic energy of  $KE = (1/2)mv^2$ . To stop the locomotive we have to remove this amount of energy. To start a shaft with moment of inertia  $I$  rotating at an angular velocity  $\omega$ , we need to provide a kinetic energy of  $KE = (1/2)I\omega^2$ . To stop the shaft we have to take this amount of energy back out. The shaft's moment of inertia is analogous to the locomotive's mass. What makes the locomotive hard to start or stop is its mass. What makes the shaft hard to start or stop is its moment of inertia. The moment of inertia depends not only on the mass of the shaft, but also its distribution.

The moment of inertia also plays a role in determining how much a horizontal metal beam will bend under a load. The stiffness of the beam is a constant times  $I$ , the moment of inertia of a typical cross-section of the beam about the beam's longitudinal axis. The greater the value of  $I$ , the stiffer the beam and the less it will bend under a given load. That is why we use I-beams instead of beams whose cross-sections are square. The flanges at the top and bottom of the beam hold most of the beam's mass away from the longitudinal axis to maximize the value of  $I$  (Figure 15.19).



**FIGURE 15.19** The greater the polar moment of inertia of the cross-section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-sectional area, but A is stiffer.

To see the moment of inertia at work, try the following experiment. Tape two coins to the ends of a pencil and twiddle the pencil about the center of mass. The moment of inertia accounts for the resistance you feel each time you change the direction of motion. Now move the coins an equal distance toward the center of mass and twiddle the pencil again. The system has the same mass and the same center of mass but now offers less resistance to the changes in motion. The moment of inertia has been reduced. The moment of inertia is what gives a baseball bat, golf club, or tennis racket its “feel.” Tennis rackets that weigh the same, look the same, and have identical centers of mass will feel different and behave differently if their masses are not distributed the same way.

Computations of moments of inertia for thin plates in the plane lead to double integral formulas, which are summarized in Table 15.2. A small thin piece of mass  $\Delta m$  is equal to its small area  $\Delta A$  multiplied by the density of a point in the piece. Computations of moments of inertia for objects occupying a region in space are discussed in Section 15.5.

The mathematical difference between the **first moments**  $M_x$  and  $M_y$  and the **moments of inertia**, or **second moments**,  $I_x$  and  $I_y$  is that the second moments use the *squares* of the “lever-arm” distances  $x$  and  $y$ .

The moment  $I_0$  is also called the **polar moment** of inertia about the origin. It is calculated by integrating the density  $\delta(x, y)$  (mass per unit area) times  $r^2 = x^2 + y^2$ , the square of the distance from a representative point  $(x, y)$  to the origin. Notice that  $I_0 = I_x + I_y$ ; once we find two, we get the third automatically. (The moment  $I_0$  is sometimes called  $I_z$ , for

moment of inertia about the  $z$ -axis. The identity  $I_z = I_x + I_y$  is then called the **Perpendicular Axis Theorem**.)

The **radius of gyration**  $R_x$  is defined by the equation

$$I_x = MR_x^2.$$

It tells how far from the  $x$ -axis the entire mass of the plate might be concentrated to give the same  $I_x$ . The radius of gyration gives a convenient way to express the moment of inertia in terms of a mass and a length. The radii  $R_y$  and  $R_0$  are defined in a similar way, with

$$I_y = MR_y^2 \quad \text{and} \quad I_0 = MR_0^2.$$

We take square roots to get the formulas in Table 15.2, which gives the formulas for moments of inertia (second moments) as well as for radii of gyration.

**TABLE 15.2** Second moment formulas for thin plates in the  $xy$ -plane

**Moments of inertia (second moments):**

About the  $x$ -axis:  $I_x = \iint y^2 \delta(x, y) \, dA$

About the  $y$ -axis:  $I_y = \iint x^2 \delta(x, y) \, dA$

About a line  $L$ :  $I_L = \iint r^2(x, y) \delta(x, y) \, dA,$   
 where  $r(x, y) =$  distance from  $(x, y)$  to  $L$

About the origin (polar moment):  $I_0 = \iint (x^2 + y^2) \delta(x, y) \, dA = I_x + I_y$

**Radii of gyration:** About the  $x$ -axis:  $R_x = \sqrt{I_x/M}$   
 About the  $y$ -axis:  $R_y = \sqrt{I_y/M}$   
 About the origin:  $R_0 = \sqrt{I_0/M}$

**EXAMPLE 5** Finding Moments of Inertia and Radii of Gyration

For the thin plate in Example 4 (Figure 15.17), find the moments of inertia and radii of gyration about the coordinate axes and the origin.

**Solution** Using the density function  $\delta(x, y) = 6x + 6y + 6$  given in Example 4, the moment of inertia about the  $x$ -axis is

$$\begin{aligned} I_x &= \int_0^1 \int_0^{2x} y^2 \delta(x, y) \, dy \, dx = \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2) \, dy \, dx \\ &= \int_0^1 \left[ 2xy^3 + \frac{3}{2}y^4 + 2y^3 \right]_{y=0}^{y=2x} \, dx = \int_0^1 (40x^4 + 16x^3) \, dx \\ &= [8x^5 + 4x^4]_0^1 = 12. \end{aligned}$$

Similarly, the moment of inertia about the  $y$ -axis is

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x, y) dy dx = \frac{39}{5}.$$

Notice that we integrate  $y^2$  times density in calculating  $I_x$  and  $x^2$  times density to find  $I_y$ .

Since we know  $I_x$  and  $I_y$ , we do not need to evaluate an integral to find  $I_0$ ; we can use the equation  $I_0 = I_x + I_y$  instead:

$$I_0 = 12 + \frac{39}{5} = \frac{60 + 39}{5} = \frac{99}{5}.$$

The three radii of gyration are

$$R_x = \sqrt{I_x/M} = \sqrt{12/14} = \sqrt{6/7} \approx 0.93$$

$$R_y = \sqrt{I_y/M} = \sqrt{\left(\frac{39}{5}\right)/14} = \sqrt{39/70} \approx 0.75$$

$$R_0 = \sqrt{I_0/M} = \sqrt{\left(\frac{99}{5}\right)/14} = \sqrt{99/70} \approx 1.19. \quad \blacksquare$$

Moments are also of importance in statistics. The first moment is used in computing the mean  $\mu$  of a set of data, and the second moment is used in computing the variance ( $\Sigma^2$ ) and the standard deviation ( $\Sigma$ ). Third and fourth moments are used for computing statistical quantities known as skewness and kurtosis.

### Centroids of Geometric Figures

When the density of an object is constant, it cancels out of the numerator and denominator of the formulas for  $\bar{x}$  and  $\bar{y}$  in Table 15.1. As far as  $\bar{x}$  and  $\bar{y}$  are concerned,  $\delta$  might as well be 1. Thus, when  $\delta$  is constant, the location of the center of mass becomes a feature of the object's shape and not of the material of which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape. To find a centroid, we set  $\delta$  equal to 1 and proceed to find  $\bar{x}$  and  $\bar{y}$  as before, by dividing first moments by masses.

#### EXAMPLE 6 Finding the Centroid of a Region

Find the centroid of the region in the first quadrant that is bounded above by the line  $y = x$  and below by the parabola  $y = x^2$ .

**Solution** We sketch the region and include enough detail to determine the limits of integration (Figure 15.20). We then set  $\delta$  equal to 1 and evaluate the appropriate formulas from Table 15.1:

$$M = \int_0^1 \int_{x^2}^x 1 dy dx = \int_0^1 \left[ y \right]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$\begin{aligned} M_x &= \int_0^1 \int_{x^2}^x y dy dx = \int_0^1 \left[ \frac{y^2}{2} \right]_{y=x^2}^{y=x} dx \\ &= \int_0^1 \left( \frac{x^2}{2} - \frac{x^4}{2} \right) dx = \left[ \frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15} \end{aligned}$$

$$M_y = \int_0^1 \int_{x^2}^x x dy dx = \int_0^1 \left[ xy \right]_{y=x^2}^{y=x} dx = \int_0^1 (x^2 - x^3) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}.$$

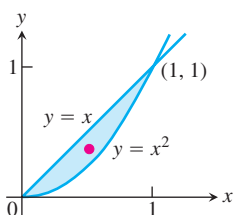


FIGURE 15.20 The centroid of this region is found in Example 6.



From these values of  $M$ ,  $M_x$ , and  $M_y$ , we find

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$

The centroid is the point  $(1/2, 2/5)$ . ■