

15.3 Double Integrals in Polar Form

Integrals are sometimes easier to evaluate if we change to polar coordinates. This section shows how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by polar equations.

Integrals in Polar Coordinates

When we defined the double integral of a function over a region R in the xy -plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have either constant x -values or constant y -values. In polar coordinates, the natural shape is a “polar rectangle” whose sides have constant r - and θ -values.

Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ for every value of θ between α and β . Then R lies in a fan-shaped region Q defined by the inequalities $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$. See Figure 15.21.

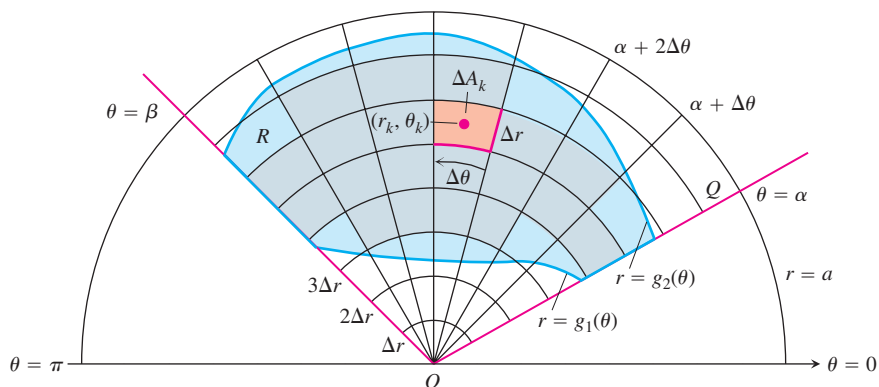


FIGURE 15.21 The region $R: g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta$, is contained in the fan-shaped region $Q: 0 \leq r \leq a, \alpha \leq \theta \leq \beta$. The partition of Q by circular arcs and rays induces a partition of R .

We cover Q by a grid of circular arcs and rays. The arcs are cut from circles centered at the origin, with radii $\Delta r, 2\Delta r, \dots, m\Delta r$, where $\Delta r = a/m$. The rays are given by

$$\theta = \alpha, \quad \theta = \alpha + \Delta\theta, \quad \theta = \alpha + 2\Delta\theta, \quad \dots, \quad \theta = \alpha + m'\Delta\theta = \beta,$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into small patches called “polar rectangles.”

We number the polar rectangles that lie inside R (the order does not matter), calling their areas $\Delta A_1, \Delta A_2, \dots, \Delta A_n$. We let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k . We then form the sum

$$S_n = \sum_{k=1}^n f(r_k, \theta_k) \Delta A_k.$$

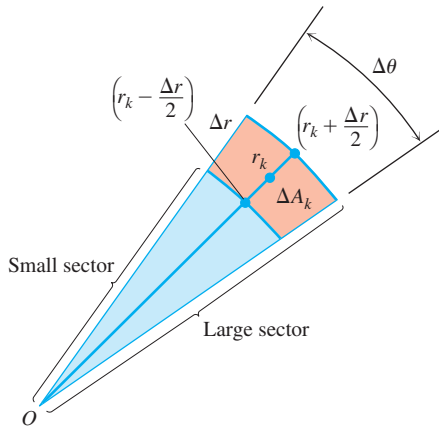


FIGURE 15.22 The observation that

$$\Delta A_k = \left(\begin{array}{c} \text{area of} \\ \text{large sector} \end{array} \right) - \left(\begin{array}{c} \text{area of} \\ \text{small sector} \end{array} \right)$$

leads to the formula $\Delta A_k = r_k \Delta r \Delta \theta$.

If f is continuous throughout R , this sum will approach a limit as we refine the grid to make Δr and $\Delta \theta$ go to zero. The limit is called the double integral of f over R . In symbols,

$$\lim_{n \rightarrow \infty} S_n = \iint_R f(r, \theta) dA.$$

To evaluate this limit, we first have to write the sum S_n in a way that expresses ΔA_k in terms of Δr and $\Delta \theta$. For convenience we choose r_k to be the average of the radii of the inner and outer arcs bounding the k th polar rectangle ΔA_k . The radius of the inner arc bounding ΔA_k is then $r_k - (\Delta r/2)$ (Figure 15.22). The radius of the outer arc is $r_k + (\Delta r/2)$.

The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$A = \frac{1}{2} \theta \cdot r^2,$$

as can be seen by multiplying πr^2 , the area of the circle, by $\theta/2\pi$, the fraction of the circle's area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are

$$\text{Inner radius: } \frac{1}{2} \left(r_k - \frac{\Delta r}{2} \right)^2 \Delta \theta$$

$$\text{Outer radius: } \frac{1}{2} \left(r_k + \frac{\Delta r}{2} \right)^2 \Delta \theta.$$

Therefore,

$$\begin{aligned} \Delta A_k &= \text{area of large sector} - \text{area of small sector} \\ &= \frac{\Delta \theta}{2} \left[\left(r_k + \frac{\Delta r}{2} \right)^2 - \left(r_k - \frac{\Delta r}{2} \right)^2 \right] = \frac{\Delta \theta}{2} (2r_k \Delta r) = r_k \Delta r \Delta \theta. \end{aligned}$$

Combining this result with the sum defining S_n gives

$$S_n = \sum_{k=1}^n f(r_k, \theta_k) r_k \Delta r \Delta \theta.$$

As $n \rightarrow \infty$ and the values of Δr and $\Delta \theta$ approach zero, these sums converge to the double integral

$$\lim_{n \rightarrow \infty} S_n = \iint_R f(r, \theta) r dr d\theta.$$

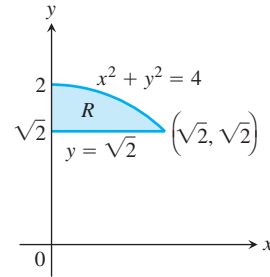
A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint_R f(r, \theta) dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta) r dr d\theta.$$

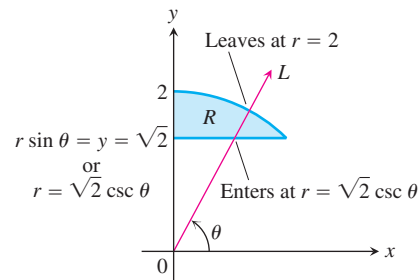
Finding Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r, \theta) dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ , take the following steps.

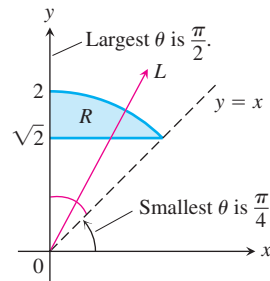
1. *Sketch:* Sketch the region and label the bounding curves.



2. *Find the r -limits of integration:* Imagine a ray L from the origin cutting through R in the direction of increasing r . Mark the r -values where L enters and leaves R . These are the r -limits of integration. They usually depend on the angle θ that L makes with the positive x -axis.



3. *Find the θ -limits of integration:* Find the smallest and largest θ -values that bound R . These are the θ -limits of integration.



The integral is

$$\iint_R f(r, \theta) \, dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2} \csc \theta}^{r=2} f(r, \theta) \, r \, dr \, d\theta.$$

EXAMPLE 1 Finding Limits of Integration

Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure 15.23).
2. Next we find the r -limits of integration. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos \theta$.

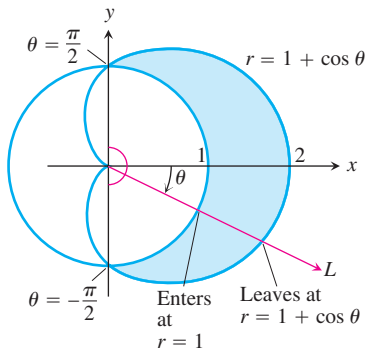


FIGURE 15.23 Finding the limits of integration in polar coordinates for the region in Example 1.

3. Finally we find the θ -limits of integration. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} f(r, \theta) r \, dr \, d\theta. \quad \blacksquare$$

If $f(r, \theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R .

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r \, dr \, d\theta.$$

This formula for area is consistent with all earlier formulas, although we do not prove this fact.

EXAMPLE 2 Finding Area in Polar Coordinates

Find the area enclosed by the lemniscate $r^2 = 4 \cos 2\theta$.

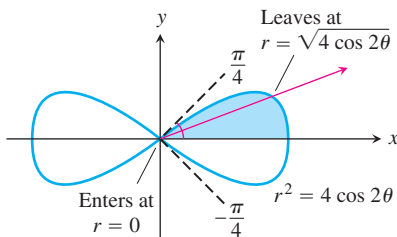


FIGURE 15.24 To integrate over the shaded region, we run r from 0 to $\sqrt{4 \cos 2\theta}$ and θ from 0 to $\pi/4$ (Example 2).

Solution We graph the lemniscate to determine the limits of integration (Figure 15.24) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$\begin{aligned} A &= 4 \int_0^{\pi/4} \int_0^{\sqrt{4 \cos 2\theta}} r \, dr \, d\theta = 4 \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_{r=0}^{r=\sqrt{4 \cos 2\theta}} d\theta \\ &= 4 \int_0^{\pi/4} 2 \cos 2\theta \, d\theta = 4 \sin 2\theta \Big|_0^{\pi/4} = 4. \quad \blacksquare \end{aligned}$$

Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_R f(x, y) \, dx \, dy$ into a polar integral has two steps. First substitute $x = r \cos \theta$ and $y = r \sin \theta$, and replace $dx \, dy$ by $r \, dr \, d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of R .

The Cartesian integral then becomes

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(r \cos \theta, r \sin \theta) r \, dr \, d\theta,$$

where G denotes the region of integration in polar coordinates. This is like the substitution method in Chapter 5 except that there are now two variables to substitute for instead of one. Notice that $dx \, dy$ is not replaced by $dr \, d\theta$ but by $r \, dr \, d\theta$. A more general discussion of changes of variables (substitutions) in multiple integrals is given in Section 15.7.

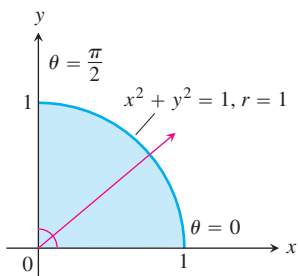


FIGURE 15.25 In polar coordinates, this region is described by simple inequalities:

$$0 \leq r \leq 1 \quad \text{and} \quad 0 \leq \theta \leq \pi/2$$

(Example 3).

EXAMPLE 3 Changing Cartesian Integrals to Polar Integrals

Find the polar moment of inertia about the origin of a thin plate of density $\delta(x, y) = 1$ bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Solution We sketch the plate to determine the limits of integration (Figure 15.25). In Cartesian coordinates, the polar moment is the value of the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx.$$

Integration with respect to y gives

$$\int_0^1 \left(x^2 \sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.

Things go better if we change the original integral to polar coordinates. Substituting $x = r \cos \theta$, $y = r \sin \theta$ and replacing $dx \, dy$ by $r \, dr \, d\theta$, we get

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx &= \int_0^{\pi/2} \int_0^1 (r^2) r \, dr \, d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{\pi}{8}. \end{aligned}$$

Why is the polar coordinate transformation so effective here? One reason is that $x^2 + y^2$ simplifies to r^2 . Another is that the limits of integration become constants. ■

EXAMPLE 4 Evaluating Integrals Using Polar Coordinates

Evaluate

$$\iint_R e^{x^2+y^2} \, dy \, dx,$$

where R is the semicircular region bounded by the x -axis and the curve $y = \sqrt{1-x^2}$ (Figure 15.26).

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y . Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to find a way to evaluate it. Polar coordinates save the day. Substituting $x = r \cos \theta$, $y = r \sin \theta$ and replacing $dy \, dx$ by $r \, dr \, d\theta$ enables us to evaluate the integral as

$$\begin{aligned} \iint_R e^{x^2+y^2} \, dy \, dx &= \int_0^{\pi} \int_0^1 e^{r^2} r \, dr \, d\theta = \int_0^{\pi} \left[\frac{1}{2} e^{r^2} \right]_0^1 d\theta \\ &= \int_0^{\pi} \frac{1}{2} (e - 1) \, d\theta = \frac{\pi}{2} (e - 1). \end{aligned}$$

The r in the $r \, dr \, d\theta$ was just what we needed to integrate e^{r^2} . Without it, we would have been unable to proceed. ■

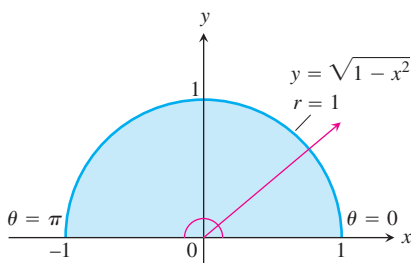


FIGURE 15.26 The semicircular region in Example 4 is the region

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi.$$