## 15.4    Triple Integrals in Rectangular Coordinates

Just as double integrals allow us to deal with more general situations than could be handled by single integrals, triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes, the masses and moments of solids of varying density, and the average value of a function over a three-dimensional region. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions, as we will see in Chapter 16.

### Triple Integrals

If $F(x, y, z)$ is a function defined on a closed bounded region $D$ in space, such as the region occupied by a solid ball or a lump of clay, then the integral of $F$ over $D$ may be defined in
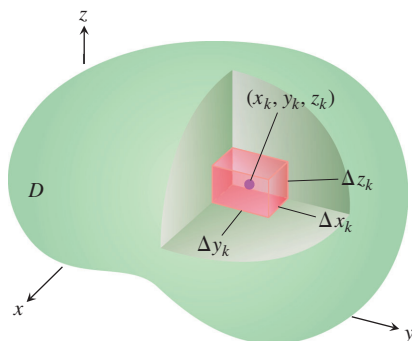
**FIGURE 15.27** Partitioning a solid with rectangular cells of volume $\Delta V_k$.

the following way. We partition a rectangular boxlike region containing $D$ into rectangular cells by planes parallel to the coordinate axis (Figure 15.27). We number the cells that lie inside $D$ from 1 to $n$ in some order, the $k$th cell having dimensions $\Delta x_k$ by $\Delta y_k$ by $\Delta z_k$ and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point $(x_k, y_k, z_k)$ in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k) \, \Delta V_k. \tag{1}$$

We are interested in what happens as $D$ is partitioned by smaller and smaller cells, so that $\Delta x_k, \Delta y_k, \Delta z_k$ and the norm of the partition $\|P\|$, the largest value among $\Delta x_k, \Delta y_k, \Delta z_k$, all approach zero. When a single limiting value is attained, no matter how the partitions and points $(x_k, y_k, z_k)$ are chosen, we say that $F$ is **integrable** over $D$. As before, it can be shown that when $F$ is continuous and the bounding surface of $D$ is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then $F$ is integrable. As $\|P\| \to 0$ and the number of cells $n$ goes to $\infty$, the sums $S_n$ approach a limit. We call this limit the **triple integral of $F$ over $D$** and write

$$\lim_{n \to \infty} S_n = \iiint_D F(x, y, z) \, dV \qquad \text{or} \qquad \lim_{\|P\| \to 0} S_n = \iiint_D F(x, y, z) \, dx \, dy \, dz.$$

The regions $D$ over which continuous functions are integrable are those that can be closely approximated by small rectangular cells. Such regions include those encountered in applications.

## Volume of a Region in Space

If $F$ is the constant function whose value is 1, then the sums in Equation (1) reduce to

$$S_n = \sum F(x_k, y_k, z_k) \, \Delta V_k = \sum 1 \cdot \Delta V_k = \sum \Delta V_k.$$

As $\Delta x_k, \Delta y_k$, and $\Delta z_k$ approach zero, the cells $\Delta V_k$ become smaller and more numerous and fill up more and more of $D$. We therefore define the volume of $D$ to be the triple integral

$$\lim_{n \to \infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV.$$

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**DEFINITION**  **Volume**
The **volume** of a closed, bounded region $D$ in space is

$$V = \iiint_D dV.$$

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This definition is in agreement with our previous definitions of volume, though we omit the verification of this fact. As we see in a moment, this integral enables us to calculate the volumes of solids enclosed by curved surfaces.
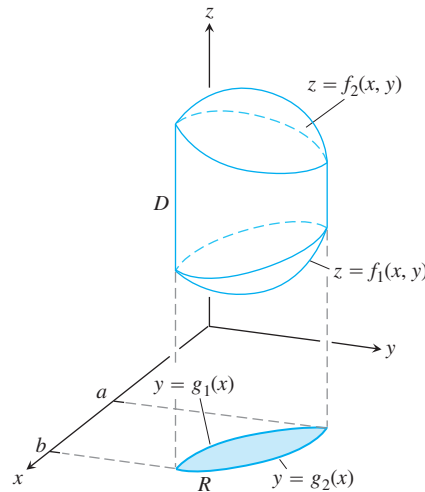
### Finding Limits of Integration

We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem (Section 15.1) to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these single integrals.

To evaluate

$$\iiint_D F(x, y, z)\, dV$$

over a region $D$, integrate first with respect to $z$, then with respect to $y$, finally with $x$.

1.  *Sketch:* Sketch the region $D$ along with its "shadow" $R$ (vertical projection) in the $xy$-plane. Label the upper and lower bounding surfaces of $D$ and the upper and lower bounding curves of $R$.



2.  *Find the z-limits of integration:* Draw a line $M$ passing through a typical point $(x, y)$ in $R$ parallel to the $z$-axis. As $z$ increases, $M$ enters $D$ at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the $z$-limits of integration.
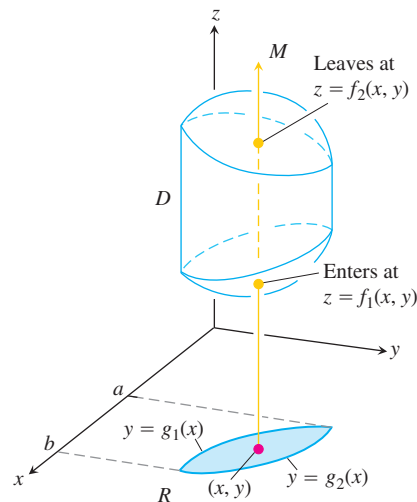
3. *Find the y-limits of integration:* Draw a line $L$ through $(x, y)$ parallel to the $y$-axis. As $y$ increases, $L$ enters $R$ at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the $y$-limits of integration.
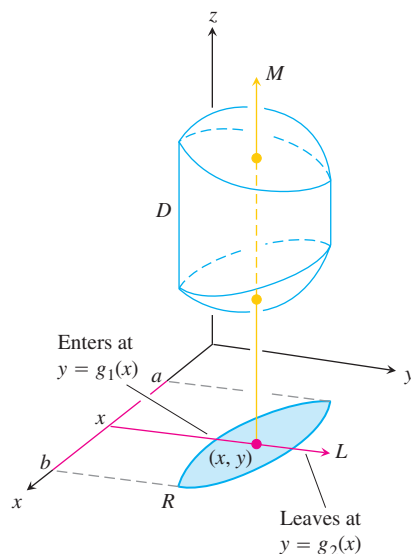


4. *Find the x-limits of integration:* Choose $x$-limits that include all lines through $R$ parallel to the $y$-axis ($x = a$ and $x = b$ in the preceding figure). These are the $x$-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z) \, dz \, dy \, dx.$$

Follow similar procedures if you change the order of integration. The "shadow" of region $D$ lies in the plane of the last two variables with respect to which the iterated integration takes place.

The above procedure applies whenever a solid region $D$ is bounded above and below by a surface, and when the "shadow" region $R$ is bounded by a lower and upper curve. It does not apply to regions with complicated holes through them, although sometimes such regions can be subdivided into simpler regions for which the procedure does apply.

### EXAMPLE 1    Finding a Volume

Find the volume of the region $D$ enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

**Solution**    The volume is

$$V = \iiint_D dz \, dy \, dx,$$

the integral of $F(x, y, z) = 1$ over $D$. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 15.28) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region $R$, the projection of $D$ onto the $xy$-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of $R$ is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.
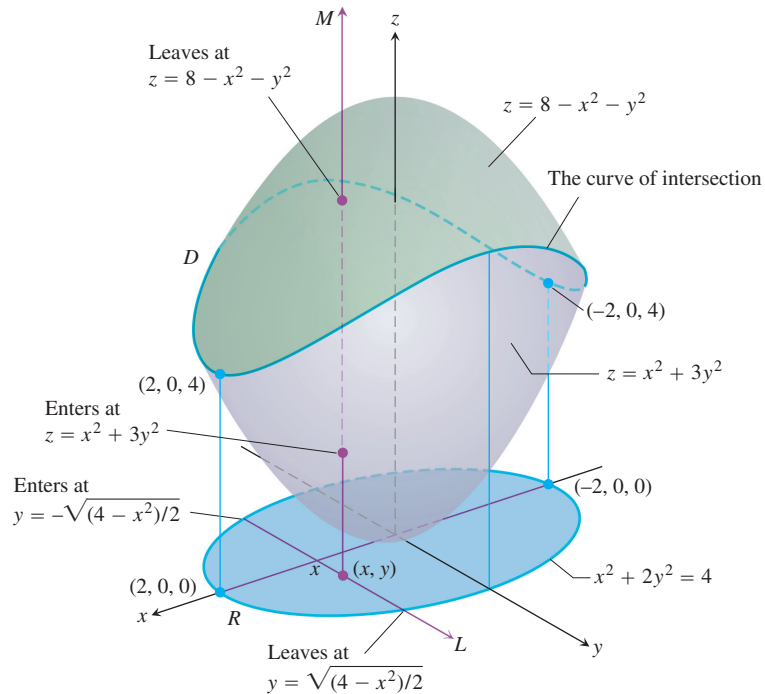
**FIGURE 15.28** The volume of the region enclosed by two paraboloids, calculated in Example 1.

Now we find the $z$-limits of integration. The line $M$ passing through a typical point $(x, y)$ in $R$ parallel to the $z$-axis enters $D$ at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

Next we find the $y$-limits of integration. The line $L$ through $(x, y)$ parallel to the $y$-axis enters $R$ at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the $x$-limits of integration. As $L$ sweeps across $R$, the value of $x$ varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of $D$ is

$$V = \iiint_D dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left[ (8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2} \left( 2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right) dx$$

$$= \int_{-2}^{2} \left[ 8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\, dx$$

$$= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2 \sin u\,. \qquad \blacksquare$$

In the next example, we project $D$ onto the $xz$-plane instead of the $xy$-plane, to show how to use a different order of integration.

### EXAMPLE 2    Finding the Limits of Integration in the Order *dy dz dx*

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron $D$ with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$.

**Solution**    We sketch $D$ along with its "shadow" $R$ in the $xz$-plane (Figure 15.29). The upper (right-hand) bounding surface of $D$ lies in the plane $y = 1$. The lower (left-hand) bounding surface lies in the plane $y = x + z$. The upper boundary of $R$ is the line $z = 1 - x$. The lower boundary is the line $z = 0$.

First we find the $y$-limits of integration. The line through a typical point $(x, z)$ in $R$ parallel to the $y$-axis enters $D$ at $y = x + z$ and leaves at $y = 1$.

Next we find the $z$-limits of integration. The line $L$ through $(x, z)$ parallel to the $z$-axis enters $R$ at $z = 0$ and leaves at $z = 1 - x$.

Finally we find the $x$-limits of integration. As $L$ sweeps across $R$, the value of $x$ varies from $x = 0$ to $x = 1$. The integral is

$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z) \, dy \, dz \, dx. \qquad \blacksquare$$



**FIGURE 15.29**   Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron $D$ (Example 2).

### EXAMPLE 3    Revisiting Example 2 Using the Order *dz dy dx*

To integrate $F(x, y, z)$ over the tetrahedron $D$ in the order $dz \, dy \, dx$, we perform the steps in the following way.

First we find the $z$-limits of integration. A line parallel to the $z$-axis through a typical point $(x, y)$ in the $xy$-plane "shadow" enters the tetrahedron at $z = 0$ and exits through the upper plane where $z = y - x$ (Figure 15.29).
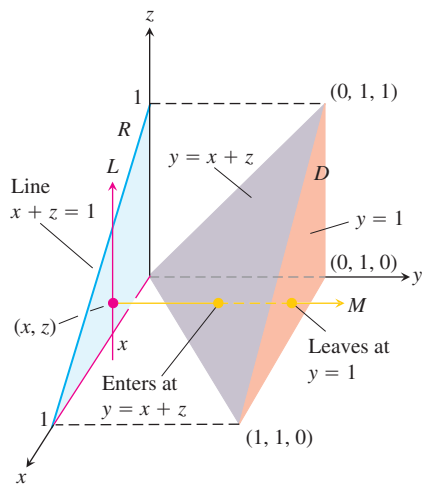
Next we find the $y$-limits of integration. On the $xy$-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line through $(x, y)$ parallel to the $y$-axis enters the shadow in the $xy$-plane at $y = x$ and exits at $y = 1$.

Finally we find the $x$-limits of integration. As the line parallel to the $y$-axis in the previous step sweeps out the shadow, the value of $x$ varies from $x = 0$ to $x = 1$ at the point $(1, 1, 0)$. The integral is

$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z) \, dz \, dy \, dx.$$

For example, if $F(x, y, z) = 1$, we would find the volume of the tetrahedron to be

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz \, dy \, dx$$

$$= \int_0^1 \int_x^1 (y - x) \, dy \, dx$$

$$= \int_0^1 \left[ \frac{1}{2} y^2 - xy \right]_{y=x}^{y=1} dx$$

$$= \int_0^1 \left( \frac{1}{2} - x + \frac{1}{2} x^2 \right) dx$$

$$= \left[ \frac{1}{2} x - \frac{1}{2} x^2 + \frac{1}{6} x^3 \right]_0^1$$

$$= \frac{1}{6}.$$

We get the same result by integrating with the order $dy\ dz\ dx$,

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\ dz\ dx = \frac{1}{6}.$$ ∎

As we have seen, there are sometimes (but not always) two different orders in which the iterated single integrations for evaluating a double integral may be worked. For triple integrals, there can be as many as six, since there are six ways of ordering $dx$, $dy$, and $dz$. Each ordering leads to a different description of the region of integration in space, and to different limits of integration.

**EXAMPLE 4**    Using Different Orders of Integration

Each of the following integrals gives the volume of the solid shown in Figure 15.30.
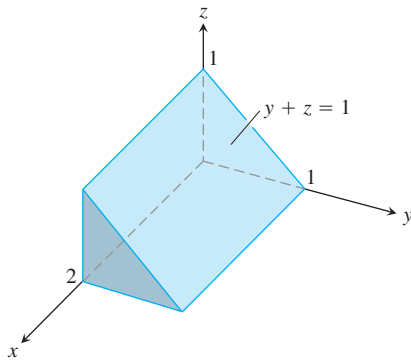


**FIGURE 15.30**  Example 4 gives six different iterated triple integrals for the volume of this prism.

**(a)** $\displaystyle\int_0^1 \int_0^{1-z} \int_0^2 dx\ dy\ dz$

**(b)** $\displaystyle\int_0^1 \int_0^{1-y} \int_0^2 dx\ dz\ dy$

**(c)** $\displaystyle\int_0^1 \int_0^2 \int_0^{1-z} dy\ dx\ dz$

**(d)** $\displaystyle\int_0^2 \int_0^1 \int_0^{1-z} dy\ dz\ dx$

**(e)** $\displaystyle\int_0^1 \int_0^2 \int_0^{1-y} dz\ dx\ dy$

**(f)** $\displaystyle\int_0^2 \int_0^1 \int_0^{1-y} dz\ dy\ dx$

We work out the integrals in parts (b) and (c):

$$V = \int_0^1 \int_0^{1-y} \int_0^2 dx\ dz\ dy \qquad \text{\color{blue}Integral in part (b)}$$

$$= \int_0^1 \int_0^{1-y} 2\ dz\ dy$$

$$= \int_0^1 \left[ 2z \right]_{z=0}^{z=1-y} dy$$

$$= \int_0^1 2(1 - y)\ dy$$

$$= 1.$$

Also,

$$V = \int_0^1 \int_0^2 \int_0^{1-z} dy\ dx\ dz \qquad \text{\color{blue}Integral in part (c)}$$

$$= \int_0^1 \int_0^2 (1 - z)\ dx\ dz$$

$$= \int_0^1 \left[ x - zx \right]_{x=0}^{x=2} dz$$

$$= \int_0^1 (2 - 2z)\, dz$$

$$= 1.$$

The integrals in parts (a), (d), (e), and (f) also give $V = 1$. ∎

## Average Value of a Function in Space

The average value of a function $F$ over a region $D$ in space is defined by the formula

$$\textbf{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\, dV. \tag{2}$$

For example, if $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, then the average value of $F$ over $D$ is the average distance of points in $D$ from the origin. If $F(x, y, z)$ is the temperature at $(x, y, z)$ on a solid that occupies a region $D$ in space, then the average value of $F$ over $D$ is the average temperature of the solid.

## EXAMPLE 5    Finding an Average Value

Find the average value of $F(x, y, z) = xyz$ over the cube bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

**Solution**    We sketch the cube with enough detail to show the limits of integration (Figure 15.31). We then use Equation (2) to calculate the average value of $F$ over the cube.

The volume of the cube is $(2)(2)(2) = 8$. The value of the integral of $F$ over the cube is

$$\int_0^2 \int_0^2 \int_0^2 xyz\, dx\, dy\, dz = \int_0^2 \int_0^2 \left[\frac{x^2}{2} yz\right]_{x=0}^{x=2} dy\, dz = \int_0^2 \int_0^2 2yz\, dy\, dz$$

$$= \int_0^2 \left[ y^2 z \right]_{y=0}^{y=2} dz = \int_0^2 4z\, dz = \left[ 2z^2 \right]_0^2 = 8.$$

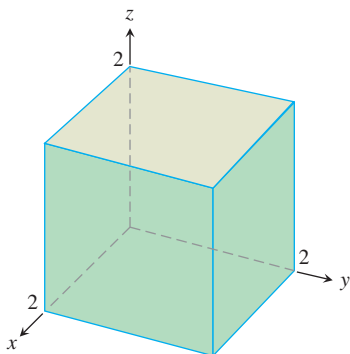With these values, Equation (2) gives

$$\begin{matrix} \text{Average value of} \\ xyz \text{ over the cube} \end{matrix} = \frac{1}{\text{volume}} \iiint_{\text{cube}} xyz\, dV = \left(\frac{1}{8}\right)(8) = 1.$$

In evaluating the integral, we chose the order $dx\, dy\, dz$, but any of the other five possible orders would have done as well. ∎

## Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals.



**FIGURE 15.31**    The region of integration in Example 5.

**Properties of Triple Integrals**

If $F = F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

1. *Constant Multiple:* $\iiint\limits_D kF \, dV = k \iiint\limits_D F \, dV$    (any number $k$)

2. *Sum and Difference:* $\iiint\limits_D (F \pm G) \, dV = \iiint\limits_D F \, dV \pm \iiint\limits_D G \, dV$

3. *Domination:*

   (a) $\iiint\limits_D F \, dV \geq 0$    if $F \geq 0$ on $D$

   (b) $\iiint\limits_D F \, dV \geq \iiint\limits_D G \, dV$    if $F \geq G$ on $D$

4. *Additivity:* $\iiint\limits_D F \, dV = \iiint\limits_{D_1} E \, dV + \iiint\limits_{D_2} F \, dV$

   if $D$ is the union of two nonoverlapping regions $D_1$ and $D_2$.