15.5 Masses and Moments in Three Dimensions

This section shows how to calculate the masses and moments of three-dimensional objects in Cartesian coordinates. The formulas are similar to those for two-dimensional objects. For calculations in spherical and cylindrical coordinates, see Section 15.6.

Masses and Moments

If $\delta(x, y, z)$ is the density of an object occupying a region D in space (mass per unit volume), the integral of δ over D gives the **mass** of the object. To see why, imagine partitioning the object into n mass elements like the one in Figure 15.32. The object's mass is the limit

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\, \Delta V_k = \iiint_D \delta(x, y, z)\, dV.$$

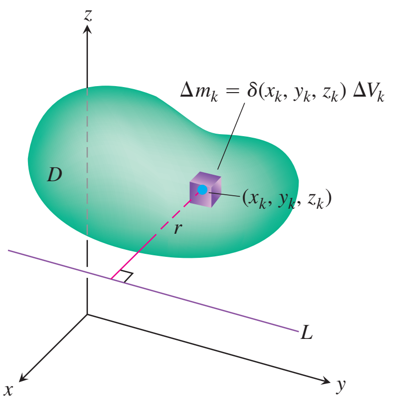

FIGURE 15.32 To define an object's mass and moment of inertia about a line, we first imagine it to be partitioned into a finite number of mass elements Δm_k.

We now derive a formula for the moment of inertia. If $r(x, y, z)$ is the distance from the point (x, y, z) in D to a line L, then the moment of inertia of the mass $\Delta m_k = \delta(x_k, y_k, z_k)\Delta V_k$ about the line L (shown in Figure 15.32) is approximately $\Delta I_k = r^2(x_k, y_k, z_k)\Delta m_k$. **The moment of inertia about *L*** of the entire object is

$$I_L = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta I_k = \lim_{n\to\infty} \sum_{k=1}^{n} r^2(x_k, y_k, z_k)\, \delta(x_k, y_k, z_k)\, \Delta V_k = \iiint_D r^2 \delta\, dV.$$

If L is the x-axis, then $r^2 = y^2 + z^2$ (Figure 15.33) and

$$I_x = \iiint_D (y^2 + z^2)\, \delta\, dV.$$

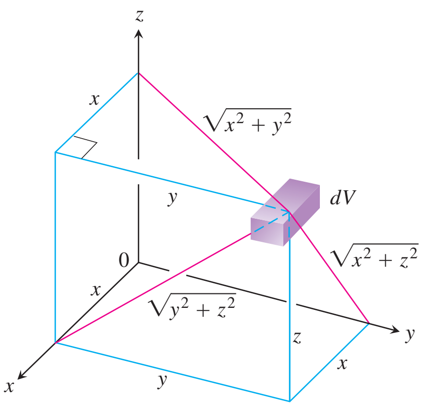

FIGURE 15.33 Distances from dV to the coordinate planes and axes.

Similarly, if L is the y-axis or z-axis we have

$$I_y = \iiint_D (x^2 + z^2)\,\delta\,dV \quad \text{and} \quad I_z = \iiint_D (x^2 + y^2)\,\delta\,dV.$$

Likewise, we can obtain the **first moments about the coordinate planes**. For example,

$$M_{yz} = \iiint_D x\delta(x, y, z)\,dV$$

gives the first moment about the yz-plane.

The mass and moment formulas in space analogous to those discussed for planar regions in Section 15.2 are summarized in Table 15.3.

TABLE 15.3 Mass and moment formulas for solid objects in space

Mass: $M = \iiint_D \delta\,dV \qquad (\delta = \delta(x, y, z) = \text{density})$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\,dV, \qquad M_{xz} = \iiint_D y\,\delta\,dV, \qquad M_{xy} = \iiint_D z\,\delta\,dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia (second moments) about the coordinate axes:

$$I_x = \iiint (y^2 + z^2)\,\delta\,dV$$

$$I_y = \iiint (x^2 + z^2)\,\delta\,dV$$

$$I_z = \iiint (x^2 + y^2)\,\delta\,dV$$

Moments of inertia about a line *L*:

$$I_L = \iiint r^2\,\delta\,dV \qquad (r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L)$$

Radius of gyration about a line *L*:

$$R_L = \sqrt{I_L/M}$$

EXAMPLE 1 Finding the Center of Mass of a Solid in Space

You Try It

Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.34).

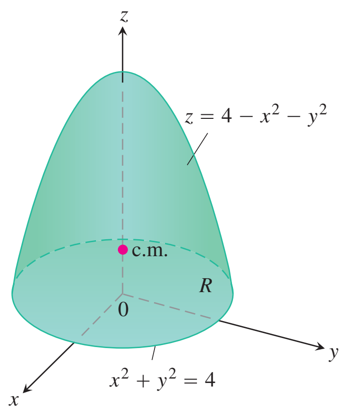

FIGURE 15.34 Finding the center of mass of a solid (Example 1).

Solution By symmetry $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we first calculate

$$M_{xy} = \iiint_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\,dz\,dy\,dx = \iint_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx$$

$$= \frac{\delta}{2}\iint_R (4 - x^2 - y^2)^2\,dy\,dx$$

$$= \frac{\delta}{2}\int_0^{2\pi}\int_0^2 (4 - r^2)^2 r\,dr\,d\theta \qquad \text{Polar coordinates}$$

$$= \frac{\delta}{2}\int_0^{2\pi}\left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3}\int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.$$

A similar calculation gives

$$M = \iiint_R \int_0^{4-x^2-y^2} \delta\,dz\,dy\,dx = 8\pi\delta.$$

Therefore $\bar{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$. ■

When the density of a solid object is constant (as in Example 1), the center of mass is called the **centroid** of the object (as was the case for two-dimensional shapes in Section 15.2).

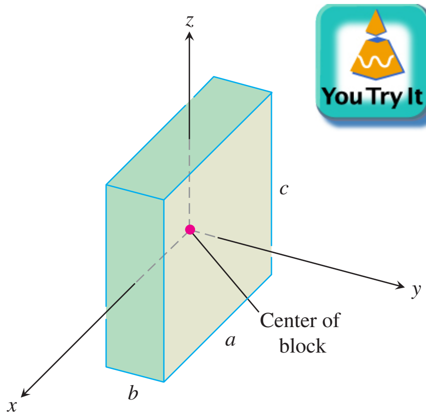

FIGURE 15.35 Finding I_x, I_y, and I_z for the block shown here. The origin lies at the center of the block (Example 2).

EXAMPLE 2 Finding the Moments of Inertia About the Coordinate Axes

Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in Figure 15.35.

Solution The formula for I_x gives

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (y^2 + z^2)\,\delta\,dx\,dy\,dz.$$

We can avoid some of the work of integration by observing that $(y^2 + z^2)\delta$ is an even function of x, y, and z. The rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$I_x = 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (y^2 + z^2)\,\delta\,dx\,dy\,dz = 4a\delta\int_0^{c/2}\int_0^{b/2} (y^2 + z^2)\,dy\,dz$$

$$= 4a\delta\int_0^{c/2}\left[\frac{y^3}{3} + z^2 y\right]_{y=0}^{y=b/2} dz$$

$$= 4a\delta\int_0^{c/2}\left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz$$

$$= 4a\delta\left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2).$$

Similarly,

$$I_y = \frac{M}{12}(a^2 + c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2 + b^2). \quad ■$$