15.6 Triple Integrals in Cylindrical and Spherical Coordinates

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this section. The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane studied in Section 15.3.

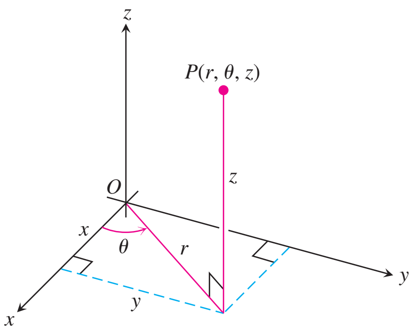


FIGURE 15.36 The cylindrical coordinates of a point in space are r, θ, and z.

Integration in Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form (r, θ, z), as shown in Figure 15.36.

DEFINITION Cylindrical Coordinates

Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

The values of $x, y, r,$ and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis (Figure 15.37). The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

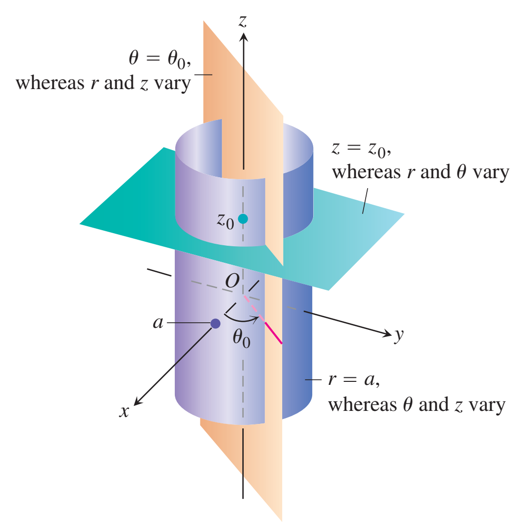


FIGURE 15.37 Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

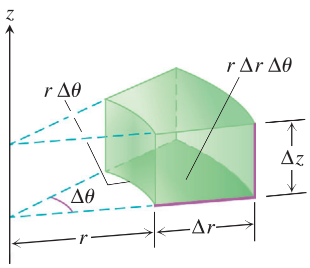


FIGURE 15.38 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

Cylindrical coordinates are good for describing cylinders whose axes run along the z-axis and planes that either contain the z-axis or lie perpendicular to the z-axis. Surfaces like these have equations of constant coordinate value:

$$r = 4. \qquad \text{Cylinder, radius 4, axis the } z\text{-axis}$$

$$\theta = \frac{\pi}{3}. \qquad \text{Plane containing the } z\text{-axis}$$

$$z = 2. \qquad \text{Plane perpendicular to the } z\text{-axis}$$

When computing triple integrals over a region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes. In the kth cylindrical wedge, r, θ and z change by $\Delta r_k, \Delta\theta_k$, and Δz_k, and the largest of these numbers among all the cylindrical wedges is called the **norm** of the partition. We define the triple integral as a limit of Riemann sums using these wedges. The volume of such a cylindrical wedge ΔV_k is obtained by taking the area ΔA_k of its base in the $r\theta$-plane and multiplying by the height Δz (Figure 15.38).

For a point (r_k, θ_k, z_k) in the center of the kth wedge, we calculated in polar coordinates that $\Delta A_k = r_k\, \Delta r_k\, \Delta\theta_k$. So $\Delta V_k = \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k$ and a Riemann sum for f over D has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k.$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero

$$\lim_{n\to\infty} S_n = \iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta.$$

Triple integrals in cylindrical coordinates are then evaluated as iterated integrals, as in the following example.

EXAMPLE 1 Finding Limits of Integration in Cylindrical Coordinates

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y - 1)^2 = 1$. Its polar coordinate equation is

$$x^2 + (y - 1)^2 = 1$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r \sin\theta = 0$$

$$r = 2 \sin\theta.$$

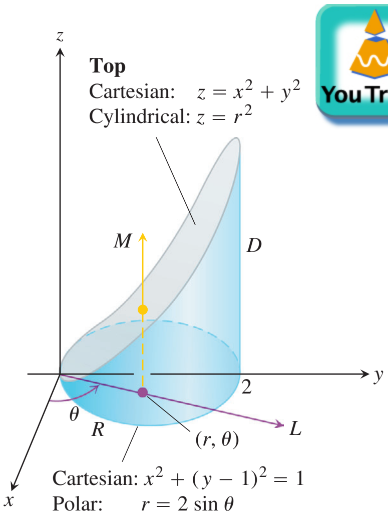


FIGURE 15.39 Finding the limits of integration for evaluating an integral in cylindrical coordinates (Example 1).

The region is sketched in Figure 15.39.

We find the limits of integration, starting with the z-limits. A line M through a typical point (r, θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2 \sin\theta$.

Finally we find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta. \quad \blacksquare$$

Example 1 illustrates a good procedure for finding limits of integration in cylindrical coordinates. The procedure is summarized as follows.

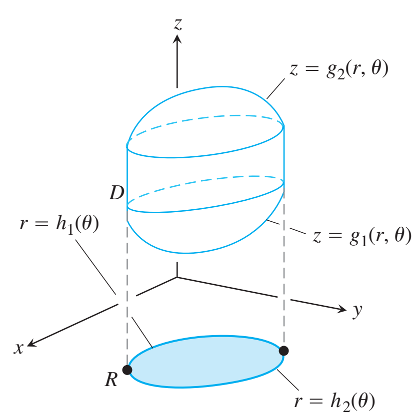
How to Integrate in Cylindrical Coordinates

To evaluate

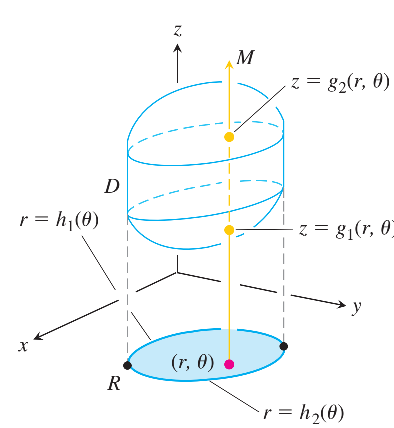
$$\iiint_D f(r, \theta, z)\, dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

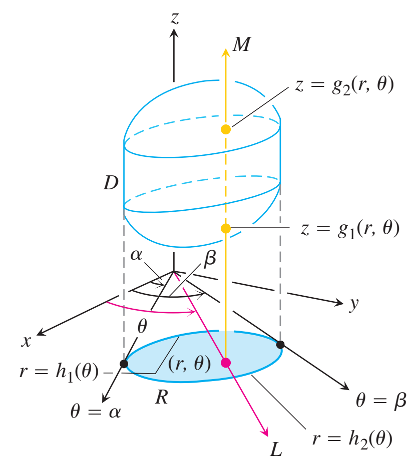
1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



2. *Find the z-limits of integration*. Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



3. *Find the r-limits of integration*. Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



4. *Find the θ-limits of integration*. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

EXAMPLE 2 Finding a Centroid

Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

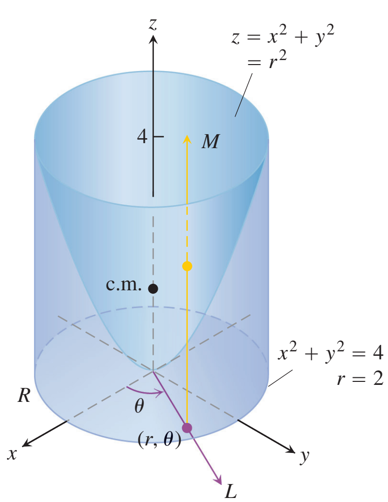


FIGURE 15.40 Example 2 shows how to find the centroid of this solid.

Solution We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (Figure 15.40). Its base R is the disk $0 \le r \le 2$ in the xy-plane.

The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

The z-limits. A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

The r-limits. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

The θ-limits. As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$. The value of M_{xy} is

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_0^{r^2} z\, dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{r^5}{2}\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi} \frac{16}{3}\, d\theta = \frac{32\pi}{3}.$$

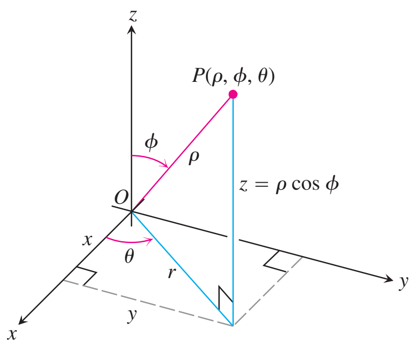


FIGURE 15.41 The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

The value of M is

$$M = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \Big[z\Big]_0^{r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4\, d\theta = 8\pi.$$

Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies outside the solid. ■

Spherical Coordinates and Integration

Spherical coordinates locate points in space with two angles and one distance, as shown in Figure 15.41. The first coordinate, $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin. Unlike r, *the variable ρ is never negative*. The second coordinate, ϕ, is the angle $\overrightarrow{OP}$ makes with the positive z-axis. It is required to lie in the interval $[0, \pi]$. The third coordinate is the angle θ as measured in cylindrical coordinates.

DEFINITION Spherical Coordinates

Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \le \phi \le \pi)$.
3. θ is the angle from cylindrical coordinates.

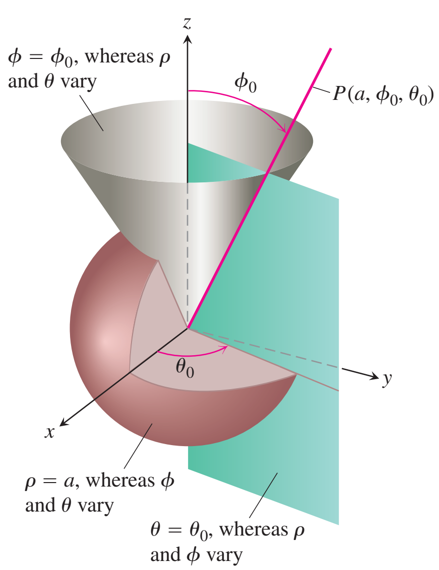


FIGURE 15.42 Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

On maps of the Earth, θ is related to the meridian of a point on the Earth and ϕ to its latitude, while ρ is related to elevation above the Earth's surface.

The equation $\rho = a$ describes the sphere of radius a centered at the origin (Figure 15.42). The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis. (We broaden our interpretation to include the xy-plane as the cone $\phi = \pi/2$.) If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward. The equation $\theta = \theta_0$ describes the half-plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho\sin\phi, \qquad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \qquad y = r\sin\theta = \rho\sin\phi\sin\theta, \tag{1}$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

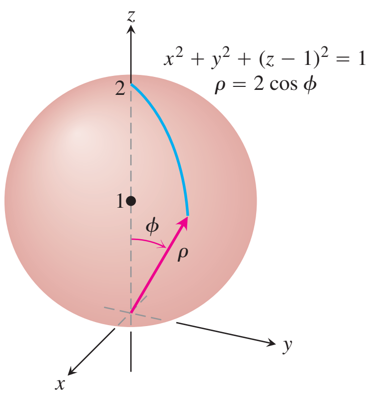


FIGURE 15.43 The sphere in Example 3.

EXAMPLE 3 Converting Cartesian to Spherical

Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.

Solution We use Equations (1) to substitute for x, y, and z:

$$x^2 + y^2 + (z - 1)^2 = 1$$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 = 1 \qquad \text{Equations (1)}$$

$$\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = 2\rho\cos\phi$$

$$\rho^2 = 2\rho\cos\phi$$

$$\rho = 2\cos\phi .$$

See Figure 15.43. ■

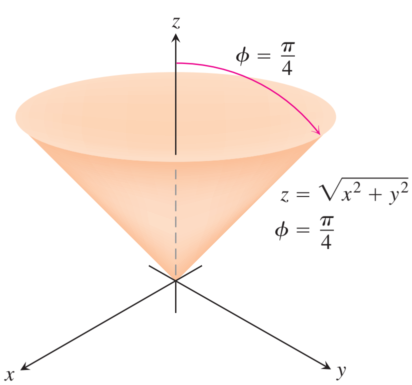


FIGURE 15.44 The cone in Example 4.

EXAMPLE 4 Converting Cartesian to Spherical

Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$ (Figure 15.44).

Solution 1 *Use geometry.* The cone is symmetric with respect to the z-axis and cuts the first quadrant of the yz-plane along the line $z = y$. The angle between the cone and the positive z-axis is therefore $\pi/4$ radians. The cone consists of the points whose spherical coordinates have ϕ equal to $\pi/4$, so its equation is $\phi = \pi/4$.

Solution 2 *Use algebra.* If we use Equations (1) to substitute for x, y, and z we obtain the same result:

$$z = \sqrt{x^2 + y^2}$$

$$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi} \qquad \text{Example 3}$$

$$\rho\cos\phi = \rho\sin\phi \qquad \rho \ge 0,\ \sin\phi \ge 0$$

$$\cos\phi = \sin\phi$$

$$\phi = \frac{\pi}{4}. \qquad 0 \le \phi \le \pi$$

■

Spherical coordinates are good for describing spheres centered at the origin, half-planes hinged along the z-axis, and cones whose vertices lie at the origin and whose axes lie along the z-axis. Surfaces like these have equations of constant coordinate value:

$$\rho = 4 \qquad \text{Sphere, radius 4, center at origin}$$

$$\phi = \frac{\pi}{3} \qquad \text{Cone opening up from the origin, making an angle of } \pi/3 \text{ radians with the positive } z\text{-axis}$$

$$\theta = \frac{\pi}{3}. \qquad \text{Half-plane, hinged along the } z\text{-axis, making an angle of } \pi/3 \text{ radians with the positive } x\text{-axis}$$

When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges. The size of the kth spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given by changes by $\Delta\rho_k$, $\Delta\theta_k$, and $\Delta\phi_k$ in ρ, θ, and ϕ. Such a spherical wedge has one edge a circular arc of length $\rho_k \,\Delta\phi_k$, another edge a circular arc of

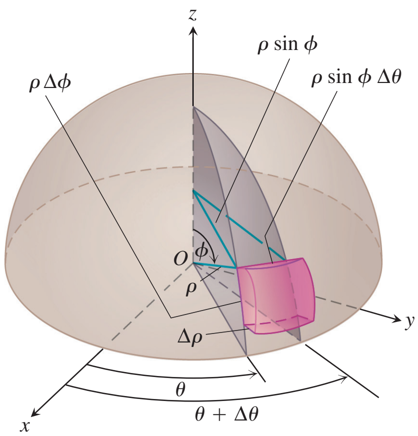


FIGURE 15.45 In spherical coordinates

$$dV = d\rho \cdot \rho \, d\phi \cdot \rho \sin \phi \, d\theta = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

length $\rho_k \sin \phi_k \, \Delta\theta_k$, and thickness $\Delta\rho_k$. The spherical wedge closely approximates a cube of these dimensions when $\Delta\rho_k$, $\Delta\theta_k$, and $\Delta\phi_k$ are all small (Figure 15.45). It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k^2 \sin \phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k$ for $(\rho_k, \phi_k, \theta_k)$ a point chosen inside the wedge.

The corresponding Riemann sum for a function $F(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} F(\rho_k, \phi_k, \theta_k) \, \rho_k^2 \sin \phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k .$$

As the norm of a partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when F is continuous:

$$\lim_{n\to\infty} S_n = \iiint_D F(\rho, \phi, \theta) \, dV = \iiint_D F(\rho, \phi, \theta) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

In spherical coordinates, we have

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ. The procedure for finding the limits of integration is shown below. We restrict our attention to integrating over domains that are solids of revolution about the z-axis (or portions thereof) and for which the limits for θ and ϕ are constant.

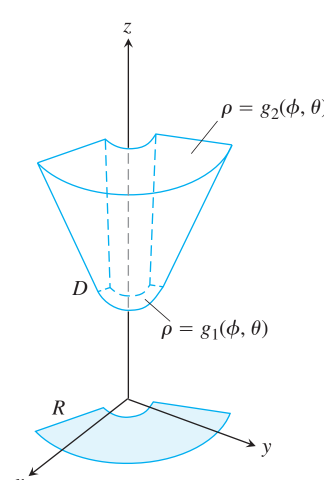
How to Integrate in Spherical Coordinates

To evaluate

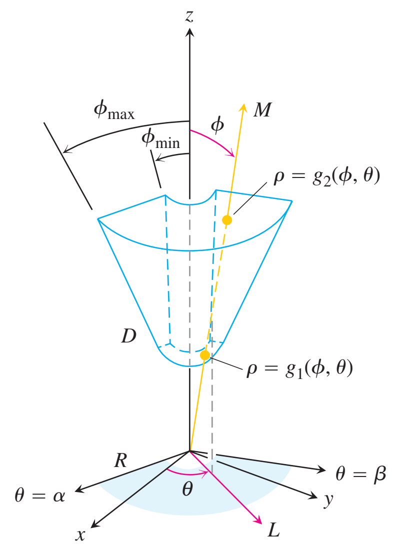
$$\iiint_D f(\rho, \phi, \theta) \, dV$$

over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.



2. *Find the ρ-limits of integration*. Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.



3. *Find the ϕ-limits of integration*. For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{min}$ to $\phi = \phi_{max}$. These are the ϕ -limits of integration.

4. *Find the θ-limits of integration*. The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

EXAMPLE 5 Finding a Volume in Spherical Coordinates

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$.

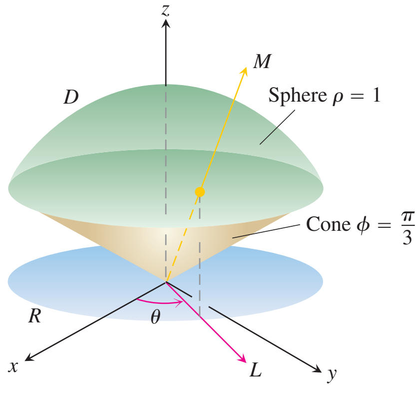


FIGURE 15.46 The ice cream cone in Example 5.

Solution The volume is $V = \iiint_D \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D.

To find the limits of integration for evaluating the integral, we begin by sketching D and its projection R on the xy-plane (Figure 15.46).

The ρ-limits of integration. We draw a ray M from the origin through D making an angle ϕ with the positive z-axis. We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

The ϕ-limits of integration. The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$.

The θ-limits of integration. The ray L sweeps over R as θ runs from 0 to 2π. The volume is

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi}\left(-\frac{1}{6}+\frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}. \quad ■$$

EXAMPLE 6 Finding a Moment of Inertia

A solid of constant density $\delta = 1$ occupies the region D in Example 5. Find the solid's moment of inertia about the z-axis.

Solution In rectangular coordinates, the moment is

$$I_z = \iiint (x^2 + y^2) \, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi$. Hence,

$$I_z = \iiint (\rho^2\sin^2\phi)\,\rho^2\sin\phi \, d\rho \, d\phi \, d\theta = \iiint \rho^4\sin^3\phi \, d\rho \, d\phi \, d\theta.$$

For the region in Example 5, this becomes

$$I_z = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4\sin^3\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi \, d\phi \, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/3}(1-\cos^2\phi)\sin\phi \, d\phi \, d\theta = \frac{1}{5}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\left(-\frac{1}{2}+1+\frac{1}{24}-\frac{1}{3}\right) d\theta = \frac{1}{5}\int_0^{2\pi}\frac{5}{24}\, d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}. \quad ■$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

In the next section we offer a more general procedure for determining dV in cylindrical and spherical coordinates. The results, of course, will be the same.