Chapter 15 Additional and Advanced Exercises

Project Project

Volumes

1. **Sand pile: double and triple integrals** The base of a sand pile covers the region in the xy-plane that is bounded by the parabola $x^2 + y = 6$ and the line $y = x$. The height of the sand above the point (x, y) is x^2. Express the volume of sand as **(a)** a double integral, **(b)** a triple integral. Then **(c)** find the volume.

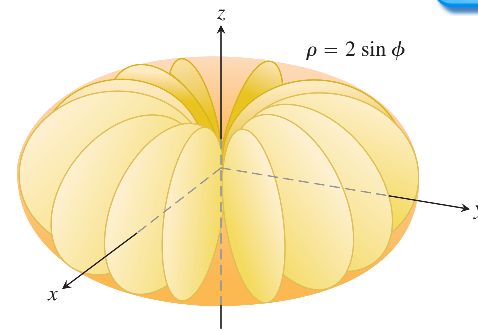
2. **Water in a hemispherical bowl** A hemispherical bowl of radius 5 cm is filled with water to within 3 cm of the top. Find the volume of water in the bowl.

3. **Solid cylindrical region between two planes** Find the volume of the portion of the solid cylinder $x^2 + y^2 \leq 1$ that lies between the planes $z = 0$ and $x + y + z = 2$.

4. **Sphere and paraboloid** Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

5. **Two paraboloids** Find the volume of the region bounded above by the paraboloid $z = 3 - x^2 - y^2$ and below by the paraboloid $z = 2x^2 + 2y^2$.

6. **Spherical coordinates** Find the volume of the region enclosed by the spherical coordinate surface $\rho = 2 \sin \phi$ (see accompanying figure).



7. **Hole in sphere** A circular cylindrical hole is bored through a solid sphere, the axis of the hole being a diameter of the sphere. The volume of the remaining solid is

$$V = 2\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta .$$

   a. Find the radius of the hole and the radius of the sphere.

   b. Evaluate the integral.

8. **Sphere and cylinder** Find the volume of material cut from the solid sphere $r^2 + z^2 \leq 9$ by the cylinder $r = 3 \sin \theta$.

9. Two paraboloids Find the volume of the region enclosed by the surfaces $z = x^2 + y^2$ and $z = (x^2 + y^2 + 1)/2$.

10. Cylinder and surface $z = xy$ Find the volume of the region in the first octant that lies between the cylinders $r = 1$ and $r = 2$ and that is bounded below by the xy-plane and above by the surface $z = xy$.

Changing the Order of Integration

11. Evaluate the integral

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}\, dx.$$

(*Hint:* Use the relation

$$\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-xy}\, dy$$

to form a double integral and evaluate the integral by changing the order of integration.)

12. a. Polar coordinates Show, by changing to polar coordinates, that

$$\int_0^{a \sin\beta} \int_{y \cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2 + y^2)\, dx\, dy = a^2\beta\left(\ln a - \frac{1}{2}\right),$$

where $a > 0$ and $0 < \beta < \pi/2$.

b. Rewrite the Cartesian integral with the order of integration reversed.

13. Reducing a double to a single integral By changing the order of integration, show that the following double integral can be reduced to a single integral:

$$\int_0^x \int_0^u e^{m(x-t)} f(t)\, dt\, du = \int_0^x (x - t)e^{m(x-t)} f(t)\, dt.$$

Similarly, it can be shown that

$$\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t)\, dt\, du\, dv = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t)\, dt.$$

14. Transforming a double integral to obtain constant limits Sometimes a multiple integral with variable limits can be changed into one with constant limits. By changing the order of integration, show that

$$\int_0^1 f(x)\left(\int_0^x g(x - y)f(y)\, dy\right) dx$$

$$= \int_0^1 f(y)\left(\int_y^1 g(x - y)f(x)\, dx\right) dy$$

$$= \frac{1}{2}\int_0^1 \int_0^1 g(|x - y|)f(x)f(y)\, dx\, dy.$$

Masses and Moments

15. Minimizing polar inertia A thin plate of constant density is to occupy the triangular region in the first quadrant of the xy-plane having vertices $(0, 0)$, $(a, 0)$, and $(a, 1/a)$. What value of a will minimize the plate's polar moment of inertia about the origin?

16. Polar inertia of triangular plate Find the polar moment of inertia about the origin of a thin triangular plate of constant density $\delta = 3$ bounded by the y-axis and the lines $y = 2x$ and $y = 4$ in the xy-plane.

17. Mass and polar inertia of a counterweight The counterweight of a flywheel of constant density 1 has the form of the smaller segment cut from a circle of radius a by a chord at a distance b from the center $(b < a)$. Find the mass of the counterweight and its polar moment of inertia about the center of the wheel.

18. Centroid of boomerang Find the centroid of the boomerang-shaped region between the parabolas $y^2 = -4(x - 1)$ and $y^2 = -2(x - 2)$ in the xy-plane.

Theory and Applications

19. Evaluate

$$\int_0^a \int_0^b e^{\max(b^2x^2,\, a^2y^2)}\, dy\, dx,$$

where a and b are positive numbers and

$$\max(b^2x^2, a^2y^2) = \begin{cases} b^2x^2 & \text{if } b^2x^2 \geq a^2y^2 \\ a^2y^2 & \text{if } b^2x^2 < a^2y^2. \end{cases}$$

20. Show that

$$\iint \frac{\partial^2 F(x, y)}{\partial x\, \partial y}\, dx\, dy$$

over the rectangle $x_0 \leq x \leq x_1$, $y_0 \leq y \leq y_1$, is

$$F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0).$$

21. Suppose that $f(x, y)$ can be written as a product $f(x, y) = F(x)G(y)$ of a function of x and a function of y. Then the integral of f over the rectangle R: $a \leq x \leq b$, $c \leq y \leq d$ can be evaluated as a product as well, by the formula

$$\iint_R f(x, y)\, dA = \left(\int_a^b F(x)\, dx\right)\left(\int_c^d G(y)\, dy\right). \qquad (1)$$

The argument is that

$$\iint_R f(x, y)\, dA = \int_c^d \left(\int_a^b F(x)G(y)\, dx\right) dy \qquad \text{(i)}$$

$$= \int_c^d \left(G(y)\int_a^b F(x)\, dx\right) dy \qquad \text{(ii)}$$

$$= \int_c^d \left(\int_a^b F(x)\, dx\right) G(y)\, dy \qquad \text{(iii)}$$

$$= \left(\int_a^b F(x)\, dx\right)\int_c^d G(y)\, dy. \qquad \text{(iv)}$$

a. Give reasons for steps (i) through (v).

When it applies, Equation (1) can be a time saver. Use it to evaluate the following integrals.

b. $\int_0^{\ln 2}\int_0^{\pi/2} e^x \cos y \, dy\, dx$ **c.** $\int_1^2\int_{-1}^1 \frac{x}{y^2}\, dx\, dy$

22. Let $D_{\mathbf{u}} f$ denote the derivative of $f(x, y) = (x^2 + y^2)/2$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$.

a. Finding average value Find the average value of $D_{\mathbf{u}} f$ over the triangular region cut from the first quadrant by the line $x + y = 1$.

b. Average value and centroid Show in general that the average value of $D_{\mathbf{u}} f$ over a region in the xy-plane is the value of $D_{\mathbf{u}} f$ at the centroid of the region.

23. The value of $\Gamma(1/2)$ The gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

extends the factorial function from the nonnegative integers to other real values. Of particular interest in the theory of differential equations is the number

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{(1/2)-1} e^{-t}\, dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt. \qquad (2)$$

a. If you have not yet done Exercise 37 in Section 15.3, do it now to show that

$$I = \int_0^\infty e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2}.$$

b. Substitute $y = \sqrt{t}$ in Equation (2) to show that $\Gamma(1/2) = 2I = \sqrt{\pi}$.

24. Total electrical charge over circular plate The electrical charge distribution on a circular plate of radius R meters is $\sigma(r, \theta) = kr(1 - \sin\theta)$ coulomb/m^2 (k a constant). Integrate σ over the plate to find the total charge Q.

25. A parabolic rain gauge A bowl is in the shape of the graph of $z = x^2 + y^2$ from $z = 0$ to $z = 10$ in. You plan to calibrate the bowl to make it into a rain gauge. What height in the bowl would correspond to 1 in. of rain? 3 in. of rain?

26. Water in a satellite dish A parabolic satellite dish is 2 m wide and 1/2 m deep. Its axis of symmetry is tilted 30 degrees from the vertical.

a. Set up, but do not evaluate, a triple integral in rectangular coordinates that gives the amount of water the satellite dish will hold. (*Hint:* Put your coordinate system so that the satellite dish is in "standard position" and the plane of the water level is slanted.) (*Caution:* The limits of integration are not "nice.")

b. What would be the smallest tilt of the satellite dish so that it holds no water?

27. An infinite half-cylinder Let D be the interior of the infinite right circular half-cylinder of radius 1 with its single-end face suspended 1 unit above the origin and its axis the ray from (0, 0, 1) to ∞. Use cylindrical coordinates to evaluate

$$\iiint_D z(r^2 + z^2)^{-5/2}\, dV.$$

28. Hypervolume We have learned that $\int_a^b 1\, dx$ is the length of the interval $[a, b]$ on the number line (one-dimensional space), $\iint_R 1\, dA$ is the area of region R in the xy-plane (two-dimensional space), and $\iiint_D 1\, dV$ is the volume of the region D in three-dimensional space (xyz-space). We could continue: If Q is a region in 4-space ($xyzw$-space), then $\iiiint_Q 1\, dV$ is the "hypervolume" of Q. Use your generalizing abilities and a Cartesian coordinate system of 4-space to find the hypervolume inside the unit 4-sphere $x^2 + y^2 + z^2 + w^2 = 1$.