# Chapter 15 Practice Exercises

## Planar Regions of Integration

In Exercises 1–4, sketch the region of integration and evaluate the double integral.

1. $\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy$

2. $\int_0^1\int_0^{x^3} e^{y/x}\,dy\,dx$

3. $\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt$

4. $\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy$

## Reversing the Order of Integration

In Exercises 5–8, sketch the region of integration and write an equivalent integral with the order of integration reversed. Then evaluate both integrals.

5. $\int_0^4\int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy$

6. $\int_0^1\int_{x^2}^{x} \sqrt{x}\,dy\,dx$

7. $\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dx\,dy$

8. $\int_0^2\int_0^{4-x^2} 2x\,dy\,dx$

## Evaluating Double Integrals

Evaluate the integrals in Exercises 9–12.

9. $\int_0^1\int_{2y}^2 4\cos(x^2)\,dx\,dy$

10. $\int_0^2\int_{y/2}^1 e^{x^2}\,dx\,dy$

11. $\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$

12. $\int_0^1\int_{\sqrt[3]{y}}^1 \frac{2\pi\sin\pi x^2}{x^2}\,dx\,dy$

## Areas and Volumes

13. **Area between line and parabola** Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the $xy$-plane.

14. **Area bounded by lines and parabola** Find the area of the "triangular" region in the $xy$-plane that is bounded on the right by the parabola $y = x^2$, on the left by the line $x + y = 2$, and above by the line $y = 4$.

15. **Volume of the region under a paraboloid** Find the volume under the paraboloid $z = x^2 + y^2$ above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the $xy$-plane.

16. **Volume of the region under parabolic cylinder** Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the $xy$-plane.

## Average Values

Find the average value of $f(x, y) = xy$ over the regions in Exercises 17 and 18.

17. The square bounded by the lines $x = 1$, $y = 1$ in the first quadrant

18. The quarter circle $x^2 + y^2 \le 1$ in the first quadrant

## Masses and Moments

19. **Centroid** Find the centroid of the "triangular" region bounded by the lines $x = 2$, $y = 2$ and the hyperbola $xy = 2$ in the $xy$-plane.

20. **Centroid** Find the centroid of the region between the parabola $x + y^2 - 2y = 0$ and the line $x + 2y = 0$ in the $xy$-plane.

21. **Polar moment** Find the polar moment of inertia about the origin of a thin triangular plate of constant density $\delta = 3$ bounded by the $y$-axis and the lines $y = 2x$ and $y = 4$ in the $xy$-plane.

22. **Polar moment** Find the polar moment of inertia about the center of a thin rectangular sheet of constant density $\delta = 1$ bounded by the lines

    **a.** $x = \pm 2$, $y = \pm 1$ in the $xy$-plane

    **b.** $x = \pm a$, $y = \pm b$ in the $xy$-plane.

    (*Hint:* Find $I_x$. Then use the formula for $I_x$ to find $I_y$ and add the two to find $I_0$).

23. **Inertial moment and radius of gyration** Find the moment of inertia and radius of gyration about the $x$-axis of a thin plate of constant density $\delta$ covering the triangle with vertices (0, 0), (3, 0), and (3, 2) in the $xy$-plane.

24. **Plate with variable density** Find the center of mass and the moments of inertia and radii of gyration about the coordinate axes of a thin plate bounded by the line $y = x$ and the parabola $y = x^2$ in the $xy$-plane if the density is $\delta(x, y) = x + 1$.

25. **Plate with variable density** Find the mass and first moments about the coordinate axes of a thin square plate bounded by the lines $x = \pm 1$, $y = \pm 1$ in the $xy$-plane if the density is $\delta(x, y) = x^2 + y^2 + 1/3$.

26. **Triangles with same inertial moment and radius of gyration** Find the moment of inertia and radius of gyration about the $x$-axis of a thin triangular plate of constant density $\delta$ whose base lies along the interval $[0, b]$ on the $x$-axis and whose vertex lies on the line $y = h$ above the $x$-axis. As you will see, it does not matter where on the line this vertex lies. All such triangles have the same moment of inertia and radius of gyration about the $x$-axis.

## Polar Coordinates

Evaluate the integrals in Exercises 27 and 28 by changing to polar coordinates.

27. $\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,dy\,dx}{(1 + x^2 + y^2)^2}$

28. $\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\,dx\,dy$

29. **Centroid** Find the centroid of the region in the polar coordinate plane defined by the inequalities $0 \le r \le 3$, $-\pi/3 \le \theta \le \pi/3$.

**30. Centroid** Find the centroid of the region in the first quadrant bounded by the rays $\theta = 0$ and $\theta = \pi/2$ and the circles $r = 1$ and $r = 3$.

**31. a. Centroid** Find the centroid of the region in the polar coordinate plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

**b.** Sketch the region and show the centroid in your sketch.

**32. a. Centroid** Find the centroid of the plane region defined by the polar coordinate inequalities $0 \le r \le a$, $-\alpha \le \theta \le \alpha$ $(0 < \alpha \le \pi)$. How does the centroid move as $\alpha \to \pi^-$?

**b.** Sketch the region for $\alpha = 5\pi/6$ and show the centroid in your sketch.

**33. Integrating over lemniscate** Integrate the function $f(x, y) = 1/(1 + x^2 + y^2)^2$ over the region enclosed by one loop of the lemniscate $(x^2 + y^2)^2 - (x^2 - y^2) = 0$.

**34.** Integrate $f(x, y) = 1/(1 + x^2 + y^2)^2$ over

**a. Triangular region** The triangle with vertices $(0, 0)$, $(1, 0)$, $(1, \sqrt{3})$.

**b. First quadrant** The first quadrant of the $xy$-plane.

## Triple Integrals in Cartesian Coordinates

Evaluate the integrals in Exercises 35–38.

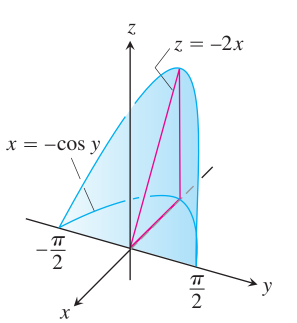
**35.** $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(x + y + z)\,dx\,dy\,dz$

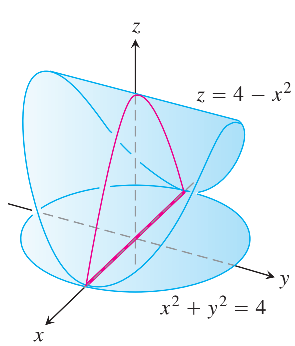
**36.** $\displaystyle\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}\,dz\,dy\,dx$

**37.** $\displaystyle\int_0^1\int_0^{x^2}\int_0^{x+y} (2x - y - z)\,dz\,dy\,dx$

**38.** $\displaystyle\int_1^e\int_1^x\int_0^z \frac{2y}{z^3}\,dy\,dz\,dx$

**39. Volume** Find the volume of the wedge-shaped region enclosed on the side by the cylinder $x = -\cos y$, $-\pi/2 \le y \le \pi/2$, on the top by the plane $z = -2x$, and below by the $xy$-plane.




**40. Volume** Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the $xy$-plane.

**41. Average value** Find the average value of $f(x, y, z) = 30xz\sqrt{x^2 + y}$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1, y = 3, z = 1$.

**42. Average value** Find the average value of $\rho$ over the solid sphere $\rho \le a$ (spherical coordinates).

## Cylindrical and Spherical Coordinates

**43. Cylindrical to rectangular coordinates** Convert

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3\,dz\,r\,dr\,d\theta, \qquad r \ge 0$$

to **(a)** rectangular coordinates with the order of integration $dz\,dx\,dy$ and **(b)** spherical coordinates. Then **(c)** evaluate one of the integrals.

**44. Rectangular to cylindrical coordinates** **(a)** Convert to cylindrical coordinates. Then **(b)** evaluate the new integral.

$$\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-(x^2+y^2)}^{(x^2+y^2)} 21xy^2\,dz\,dy\,dx$$

**45. Rectangular to spherical coordinates** **(a)** Convert to spherical coordinates. Then **(b)** evaluate the new integral.

$$\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} dz\,dy\,dx$$

**46. Rectangular, cylindrical, and spherical coordinates** Write an iterated triple integral for the integral of $f(x, y, z) = 6 + 4y$ over the region in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 1$, and the coordinate planes in **(a)** rectangular coordinates, **(b)** cylindrical coordinates, and **(c)** spherical coordinates. Then **(d)** find the integral of $f$ by evaluating one of the triple integrals.

**47. Cylindrical to rectangular coordinates** Set up an integral in rectangular coordinates equivalent to the integral

$$\int_0^{\pi/2}\int_1^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2\,dz\,dr\,d\theta.$$

Arrange the order of integration to be $z$ first, then $y$, then $x$.

**48. Rectangular to cylindrical coordinates** The volume of a solid is

$$\int_0^2\int_0^{\sqrt{2x-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx.$$

**a.** Describe the solid by giving equations for the surfaces that form its boundary.

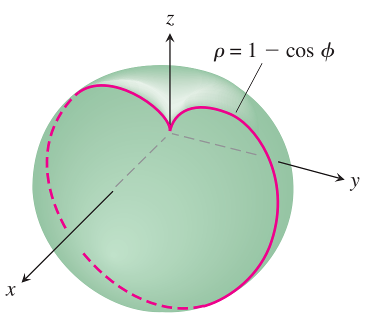
**b.** Convert the integral to cylindrical coordinates but do not evaluate the integral.

**49. Spherical versus cylindrical coordinates** Triple integrals involving spherical shapes do not always require spherical coordinates for convenient evaluation. Some calculations may be accomplished more easily with cylindrical coordinates. As a case in point, find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the plane $z = 2$ by using **(a)** cylindrical coordinates and **(b)** spherical coordinates.

**50. Finding $I_z$ in spherical coordinates** Find the moment of inertia about the $z$-axis of a solid of constant density $\delta = 1$ that is bounded above by the sphere $\rho = 2$ and below by the cone $\phi = \pi/3$ (spherical coordinates).

**51. Moment of inertia of a "thick" sphere** Find the moment of inertia of a solid of constant density $\delta$ bounded by two concentric spheres of radii $a$ and $b$ $(a < b)$ about a diameter.

**52. Moment of inertia of an apple** Find the moment of inertia about the $z$-axis of a solid of density $\delta = 1$ enclosed by the spherical coordinate surface $\rho = 1 - \cos\phi$. The solid is the red curve rotated about the $z$-axis in the accompanying figure.




## Substitutions

**53.** Show that if $u = x - y$ and $v = y$, then

$$\int_0^\infty \int_0^x e^{-sx} f(x - y, y)\, dy\, dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u, v)\, du\, dv.$$

**54.** What relationship must hold between the constants $a$, $b$, and $c$ to make

$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(ax^2+2bxy+cy^2)}\, dx\, dy = 1?$$

(*Hint:* Let $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$, where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$. Then $ax^2 + 2bxy + cy^2 = s^2 + t^2$.)