

## EXERCISES 16.2

## Vector and Gradient Fields

Find the gradient fields of the functions in Exercises 1–4.

- $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
- $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$
- $g(x, y, z) = e^z - \ln(x^2 + y^2)$
- $g(x, y, z) = xy + yz + xz$
- Give a formula  $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$  for the vector field in the plane that has the property that  $\mathbf{F}$  points toward the origin with magnitude inversely proportional to the square of the distance from  $(x, y)$  to the origin. (The field is not defined at  $(0, 0)$ .)
- Give a formula  $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$  for the vector field in the plane that has the properties that  $\mathbf{F} = \mathbf{0}$  at  $(0, 0)$  and that at any other point  $(a, b)$ ,  $\mathbf{F}$  is tangent to the circle  $x^2 + y^2 = a^2 + b^2$  and points in the clockwise direction with magnitude  $|\mathbf{F}| = \sqrt{a^2 + b^2}$ .

## Work

In Exercises 7–12, find the work done by force  $\mathbf{F}$  from  $(0, 0, 0)$  to  $(1, 1, 1)$  over each of the following paths (Figure 16.21):

- The straight-line path  $C_1$ :  $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq 1$
  - The curved path  $C_2$ :  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$ ,  $0 \leq t \leq 1$
  - The path  $C_3 \cup C_4$  consisting of the line segment from  $(0, 0, 0)$  to  $(1, 1, 0)$  followed by the segment from  $(1, 1, 0)$  to  $(1, 1, 1)$
- $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$       8.  $\mathbf{F} = [1/(x^2 + 1)]\mathbf{j}$
  - $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$       10.  $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
  - $\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$
  - $\mathbf{F} = (y + z)\mathbf{i} + (z + x)\mathbf{j} + (x + y)\mathbf{k}$

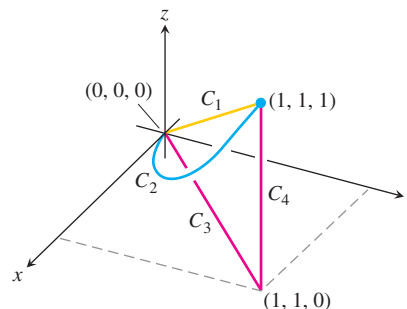


FIGURE 16.21 The paths from  $(0, 0, 0)$  to  $(1, 1, 1)$ .

In Exercises 13–16, find the work done by  $\mathbf{F}$  over the curve in the direction of increasing  $t$ .

- $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$   
 $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq 1$
- $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$   
 $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}$ ,  $0 \leq t \leq 2\pi$
- $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$   
 $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq 2\pi$
- $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$   
 $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}$ ,  $0 \leq t \leq 2\pi$

## Line Integrals and Vector Fields in the Plane

- Evaluate  $\int_C xy \, dx + (x + y) \, dy$  along the curve  $y = x^2$  from  $(-1, 1)$  to  $(2, 4)$ .
- Evaluate  $\int_C (x - y) \, dx + (x + y) \, dy$  counterclockwise around the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ .

19. Evaluate  $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$  for the vector field  $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$  along the curve  $x = y^2$  from  $(4, 2)$  to  $(1, -1)$ .

20. Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  for the vector field  $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$  counterclockwise along the unit circle  $x^2 + y^2 = 1$  from  $(1, 0)$  to  $(0, 1)$ .

21. **Work** Find the work done by the force  $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$  over the straight line from  $(1, 1)$  to  $(2, 3)$ .

22. **Work** Find the work done by the gradient of  $f(x, y) = (x + y)^2$  counterclockwise around the circle  $x^2 + y^2 = 4$  from  $(2, 0)$  to itself.

23. **Circulation and flux** Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

a. The circle  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ ,  $0 \leq t \leq 2\pi$

b. The ellipse  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$ ,  $0 \leq t \leq 2\pi$

24. **Flux across a circle** Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi.$$

## Circulation and Flux

In Exercises 25–28, find the circulation and flux of the field  $\mathbf{F}$  around and across the closed semicircular path that consists of the semicircular arch  $\mathbf{r}_1(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$ ,  $0 \leq t \leq \pi$ , followed by the line segment  $\mathbf{r}_2(t) = t\mathbf{i}$ ,  $-a \leq t \leq a$ .

25.  $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$                       26.  $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$

27.  $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$                     28.  $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$

29. **Flow integrals** Find the flow of the velocity field  $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$  along each of the following paths from  $(1, 0)$  to  $(-1, 0)$  in the  $xy$ -plane.

a. The upper half of the circle  $x^2 + y^2 = 1$

b. The line segment from  $(1, 0)$  to  $(-1, 0)$

c. The line segment from  $(1, 0)$  to  $(0, -1)$  followed by the line segment from  $(0, -1)$  to  $(-1, 0)$ .

30. **Flux across a triangle** Find the flux of the field  $\mathbf{F}$  in Exercise 29 outward across the triangle with vertices  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$ .

## Sketching and Finding Fields in the Plane

31. **Spin field** Draw the spin field

$$\mathbf{F} = -\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$$

(see Figure 16.14) along with its horizontal and vertical components at a representative assortment of points on the circle  $x^2 + y^2 = 4$ .

32. **Radial field** Draw the radial field

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$

(see Figure 16.13) along with its horizontal and vertical components at a representative assortment of points on the circle  $x^2 + y^2 = 1$ .

33. **A field of tangent vectors**

a. Find a field  $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$  in the  $xy$ -plane with the property that at any point  $(a, b) \neq (0, 0)$ ,  $\mathbf{G}$  is a vector of magnitude  $\sqrt{a^2 + b^2}$  tangent to the circle  $x^2 + y^2 = a^2 + b^2$  and pointing in the counterclockwise direction. (The field is undefined at  $(0, 0)$ .)

b. How is  $\mathbf{G}$  related to the spin field  $\mathbf{F}$  in Figure 16.14?

34. **A field of tangent vectors**

a. Find a field  $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$  in the  $xy$ -plane with the property that at any point  $(a, b) \neq (0, 0)$ ,  $\mathbf{G}$  is a unit vector tangent to the circle  $x^2 + y^2 = a^2 + b^2$  and pointing in the clockwise direction.

b. How is  $\mathbf{G}$  related to the spin field  $\mathbf{F}$  in Figure 16.14?

35. **Unit vectors pointing toward the origin** Find a field  $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$  in the  $xy$ -plane with the property that at each point  $(x, y) \neq (0, 0)$ ,  $\mathbf{F}$  is a unit vector pointing toward the origin. (The field is undefined at  $(0, 0)$ .)

36. **Two “central” fields** Find a field  $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$  in the  $xy$ -plane with the property that at each point  $(x, y) \neq (0, 0)$ ,  $\mathbf{F}$  points toward the origin and  $|\mathbf{F}|$  is (a) the distance from  $(x, y)$  to the origin, (b) inversely proportional to the distance from  $(x, y)$  to the origin. (The field is undefined at  $(0, 0)$ .)

## Flow Integrals in Space

In Exercises 37–40,  $\mathbf{F}$  is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve in the direction of increasing  $t$ .

37.  $\mathbf{F} = -4xy\mathbf{i} + 8y\mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \leq t \leq 2$$

38.  $\mathbf{F} = x^2\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$

$$\mathbf{r}(t) = 3t\mathbf{j} + 4t\mathbf{k}, \quad 0 \leq t \leq 1$$

39.  $\mathbf{F} = (x - z)\mathbf{i} + x\mathbf{k}$

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{k}, \quad 0 \leq t \leq \pi$$

40.  $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 2\mathbf{k}$

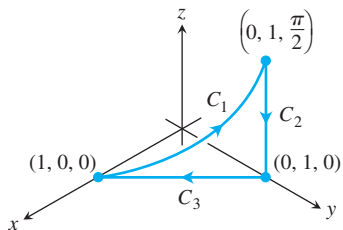
$$\mathbf{r}(t) = (-2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + 2t\mathbf{k}, \quad 0 \leq t \leq 2\pi$$

41. **Circulation** Find the circulation of  $\mathbf{F} = 2x\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$  around the closed path consisting of the following three curves traversed in the direction of increasing  $t$ :

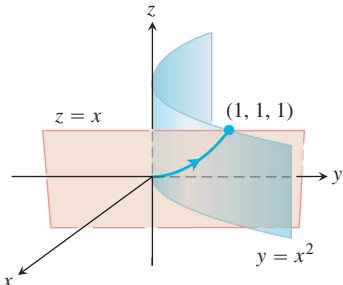
$$C_1: \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq \pi/2$$

$$C_2: \mathbf{r}(t) = \mathbf{j} + (\pi/2)(1 - t)\mathbf{k}, \quad 0 \leq t \leq 1$$

$$C_3: \mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \leq t \leq 1$$



42. **Zero circulation** Let  $C$  be the ellipse in which the plane  $2x + 3y - z = 0$  meets the cylinder  $x^2 + y^2 = 12$ . Show, without evaluating either line integral directly, that the circulation of the field  $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  around  $C$  in either direction is zero.
43. **Flow along a curve** The field  $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$  is the velocity field of a flow in space. Find the flow from  $(0, 0, 0)$  to  $(1, 1, 1)$  along the curve of intersection of the cylinder  $y = x^2$  and the plane  $z = x$ . (*Hint:* Use  $t = x$  as the parameter.)



44. **Flow of a gradient field** Find the flow of the field  $\mathbf{F} = \nabla(xy^2z^3)$ :
- Once around the curve  $C$  in Exercise 42, clockwise as viewed from above
  - Along the line segment from  $(1, 1, 1)$  to  $(2, 1, -1)$ ,

### Theory and Examples

45. **Work and area** Suppose that  $f(t)$  is differentiable and positive for  $a \leq t \leq b$ . Let  $C$  be the path  $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$ ,  $a \leq t \leq b$ , and  $\mathbf{F} = y\mathbf{i}$ . Is there any relation between the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the  $t$ -axis, the graph of  $f$ , and the lines  $t = a$  and  $t = b$ ? Give reasons for your answer.

46. **Work done by a radial force with constant magnitude** A particle moves along the smooth curve  $y = f(x)$  from  $(a, f(a))$  to  $(b, f(b))$ . The force moving the particle has constant magnitude  $k$  and always points away from the origin. Show that the work done by the force is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = k[(b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}].$$

### COMPUTER EXPLORATIONS

#### Finding Work Numerically

In Exercises 47–52, use a CAS to perform the following steps for finding the work done by force  $\mathbf{F}$  over the given path:

- Find  $d\mathbf{r}$  for the path  $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ .
  - Evaluate the force  $\mathbf{F}$  along the path.
  - Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ .
47.  $\mathbf{F} = xy^6\mathbf{i} + 3x(xy^5 + 2)\mathbf{j}$ ;  $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ ,  $0 \leq t \leq 2\pi$
48.  $\mathbf{F} = \frac{3}{1+x^2}\mathbf{i} + \frac{2}{1+y^2}\mathbf{j}$ ;  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ ,  $0 \leq t \leq \pi$
49.  $\mathbf{F} = (y + yz \cos xyz)\mathbf{i} + (x^2 + xz \cos xyz)\mathbf{j} + (z + xy \cos xyz)\mathbf{k}$ ;  $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + \mathbf{k}$ ,  $0 \leq t \leq 2\pi$
50.  $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + ze^x\mathbf{k}$ ;  $\mathbf{r}(t) = -t\mathbf{i} + \sqrt{t}\mathbf{j} + 3t\mathbf{k}$ ,  $1 \leq t \leq 4$
51.  $\mathbf{F} = (2y + \sin x)\mathbf{i} + (z^2 + (1/3)\cos y)\mathbf{j} + x^4\mathbf{k}$ ;  $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin 2t)\mathbf{k}$ ,  $-\pi/2 \leq t \leq \pi/2$
52.  $\mathbf{F} = (x^2y)\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$ ;  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (2 \sin^2 t - 1)\mathbf{k}$ ,  $0 \leq t \leq 2\pi$