

EXERCISES 16.3

Testing for Conservative Fields

Which fields in Exercises 1–6 are conservative, and which are not?

- $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
- $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
- $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} - y\mathbf{k}$
- $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
- $\mathbf{F} = (z + y)\mathbf{i} + z\mathbf{j} + (y + x)\mathbf{k}$
- $\mathbf{F} = (e^x \cos y)\mathbf{i} - (e^x \sin y)\mathbf{j} + z\mathbf{k}$

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field \mathbf{F} .

- $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$
- $\mathbf{F} = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$
- $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$
- $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
- $\mathbf{F} = (\ln x + \sec^2(x + y))\mathbf{i} + \left(\sec^2(x + y) + \frac{y}{y^2 + z^2}\right)\mathbf{j} + \frac{z}{y^2 + z^2}\mathbf{k}$
- $\mathbf{F} = \frac{y}{1 + x^2 y^2}\mathbf{i} + \left(\frac{x}{1 + x^2 y^2} + \frac{z}{\sqrt{1 - y^2 z^2}}\right)\mathbf{j} + \left(\frac{y}{\sqrt{1 - y^2 z^2}} + \frac{1}{z}\right)\mathbf{k}$

Evaluating Line Integrals

In Exercises 13–17, show that the differential forms in the integrals are exact. Then evaluate the integrals.

- $\int_{(0,0,0)}^{(2,3,-6)} 2x \, dx + 2y \, dy + 2z \, dz$
- $\int_{(1,1,2)}^{(3,5,0)} yz \, dx + xz \, dy + xy \, dz$
- $\int_{(0,0,0)}^{(1,2,3)} 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz$
- $\int_{(0,0,0)}^{(3,3,1)} 2x \, dx - y^2 \, dy - \frac{4}{1 + z^2} \, dz$
- $\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x \, dx + \cos y \sin x \, dy + dz$

Although they are not defined on all of space R^3 , the fields associated with Exercises 18–22 are simply connected and the Component Test can be used to show they are conservative. Find a potential function for each field and evaluate the integrals as in Example 4.

- $\int_{(0,2,1)}^{(1,\pi/2,2)} 2 \cos y \, dx + \left(\frac{1}{y} - 2x \sin y\right) \, dy + \frac{1}{z} \, dz$

- $\int_{(1,1,1)}^{(1,2,3)} 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln y \, dz$
- $\int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz) \, dx + \left(\frac{x^2}{y} - xz\right) \, dy - xy \, dz$
- $\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} \, dx + \left(\frac{1}{z} - \frac{x}{y^2}\right) \, dy - \frac{y}{z^2} \, dz$
- $\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x \, dx + 2y \, dy + 2z \, dz}{x^2 + y^2 + z^2}$
- Revisiting Example 4** Evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y \, dx + x \, dy + 4 \, dz$$

from Example 4 by finding parametric equations for the line segment from $(1, 1, 1)$ to $(2, 3, -1)$ and evaluating the line integral of $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + 4\mathbf{k}$ along the segment. Since \mathbf{F} is conservative, the integral is independent of the path.

- Evaluate

$$\int_C x^2 \, dx + yz \, dy + (y^2/2) \, dz$$

along the line segment C joining $(0, 0, 0)$ to $(0, 3, 4)$.

Theory, Applications, and Examples

Independence of path Show that the values of the integrals in Exercises 25 and 26 do not depend on the path taken from A to B .

- $\int_A^B z^2 \, dx + 2y \, dy + 2xz \, dz$
- $\int_A^B \frac{x \, dx + y \, dy + z \, dz}{\sqrt{x^2 + y^2 + z^2}}$

In Exercises 27 and 28, find a potential function for \mathbf{F} .

- $\mathbf{F} = \frac{2x}{y}\mathbf{i} + \left(\frac{1 - x^2}{y^2}\right)\mathbf{j}$
- $\mathbf{F} = (e^x \ln y)\mathbf{i} + \left(\frac{e^x}{y} + \sin z\right)\mathbf{j} + (y \cos z)\mathbf{k}$

- Work along different paths** Find the work done by $\mathbf{F} = (x^2 + y)\mathbf{i} + (y^2 + x)\mathbf{j} + ze^z\mathbf{k}$ over the following paths from $(1, 0, 0)$ to $(1, 0, 1)$.

- The line segment $x = 1, y = 0, 0 \leq z \leq 1$
- The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/2\pi)\mathbf{k}, 0 \leq t \leq 2\pi$
- The x -axis from $(1, 0, 0)$ to $(0, 0, 0)$ followed by the parabola $z = x^2, y = 0$ from $(0, 0, 0)$ to $(1, 0, 1)$

- Work along different paths** Find the work done by $\mathbf{F} = e^{yz}\mathbf{i} + (xze^{yz} + z \cos y)\mathbf{j} + (xye^{yz} + \sin y)\mathbf{k}$ over the following paths from $(1, 0, 1)$ to $(1, \pi/2, 0)$.

- a. The line segment $x = 1, y = \pi t/2, z = 1 - t, 0 \leq t \leq 1$
- b. The line segment from $(1, 0, 1)$ to the origin followed by the line segment from the origin to $(1, \pi/2, 0)$
- c. The line segment from $(1, 0, 1)$ to $(1, 0, 0)$, followed by the x -axis from $(1, 0, 0)$ to the origin, followed by the parabola $y = \pi x^2/2, z = 0$ from there to $(1, \pi/2, 0)$
- 31. Evaluating a work integral two ways** Let $\mathbf{F} = \nabla(x^3y^2)$ and let C be the path in the xy -plane from $(-1, 1)$ to $(1, 1)$ that consists of the line segment from $(-1, 1)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(1, 1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.
- a. Find parametrizations for the segments that make up C and evaluate the integral.
- b. Using $f(x, y) = x^3y^2$ as a potential function for \mathbf{F} .
- 32. Integral along different paths** Evaluate $\int_C 2x \cos y \, dx - x^2 \sin y \, dy$ along the following paths C in the xy -plane.
- a. The parabola $y = (x - 1)^2$ from $(1, 0)$ to $(0, 1)$
- b. The line segment from $(-1, \pi)$ to $(1, 0)$
- c. The x -axis from $(-1, 0)$ to $(1, 0)$
- d. The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, 0 \leq t \leq 2\pi$, counterclockwise from $(1, 0)$ back to $(1, 0)$
- 33. a. Exact differential form** How are the constants a, b , and c related if the following differential form is exact?
- $$(ay^2 + 2czx) \, dx + y(bx + cz) \, dy + (ay^2 + cx^2) \, dz$$
- b. **Gradient field** For what values of b and c will
- $$\mathbf{F} = (y^2 + 2czx)\mathbf{i} + y(bx + cz)\mathbf{j} + (y^2 + cx^2)\mathbf{k}$$
- be a gradient field?

- 34. Gradient of a line integral** Suppose that $\mathbf{F} = \nabla f$ is a conservative vector field and

$$g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r}.$$

Show that $\nabla g = \mathbf{F}$.

- 35. Path of least work** You have been asked to find the path along which a force field \mathbf{F} will perform the least work in moving a particle between two locations. A quick calculation on your part shows \mathbf{F} to be conservative. How should you respond? Give reasons for your answer.
- 36. A revealing experiment** By experiment, you find that a force field \mathbf{F} performs only half as much work in moving an object along path C_1 from A to B as it does in moving the object along path C_2 from A to B . What can you conclude about \mathbf{F} ? Give reasons for your answer.
- 37. Work by a constant force** Show that the work done by a constant force field $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in moving a particle along any path from A to B is $W = \mathbf{F} \cdot \overrightarrow{AB}$.
- 38. Gravitational field**

- a. Find a potential function for the gravitational field

$$\mathbf{F} = -GmM \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} \quad (G, m, \text{ and } M \text{ are constants}).$$

- b. Let P_1 and P_2 be points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM \left(\frac{1}{s_2} - \frac{1}{s_1} \right).$$