16.4 Green's Theorem in the Plane

From Table 16.2 in Section 16.2, we know that every line integral $\int_C M\,dx + N\,dy$ can be written as a flow integral $\int_a^b \mathbf{F}\cdot\mathbf{T}\,ds$. If the integral is independent of path, so the field $\mathbf{F}$ is conservative (over a domain satisfying the basic assumptions), we can evaluate the integral easily from a potential function for the field. In this section we consider how to evaluate the integral if it is *not* associated with a conservative vector field, but is a flow or flux integral across a closed curve in the xy-plane. The means for doing so is a result known as Green's Theorem, which converts the line integral into a double integral over the region enclosed by the path.

We frame our discussion in terms of velocity fields of fluid flows because they are easy to picture. However, Green's Theorem applies to any vector field satisfying certain mathematical conditions. It does not depend for its validity on the field's having a particular physical interpretation.

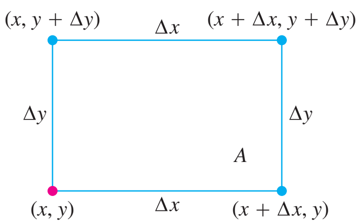

FIGURE 16.24 The rectangle for defining the divergence (flux density) of a vector field at a point (x, y).

Divergence

We need two new ideas for Green's Theorem. The first is the idea of the *divergence* of a vector field at a point, sometimes called the *flux density* of the vector field by physicists and engineers. We obtain it in the following way.

Suppose that $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is the velocity field of a fluid flow in the plane and that the first partial derivatives of M and N are continuous at each point of a region R. Let (x, y) be a point in R and let A be a small rectangle with one corner at (x, y) that, along with its interior, lies entirely in R (Figure 16.24). The sides of the rectangle, parallel to the coordinate axes, have lengths of Δx and Δy. The rate at which fluid leaves the rectangle across the bottom edge is approximately

$$\mathbf{F}(x, y) \cdot (-\mathbf{j})\, \Delta x = -N(x, y)\Delta x.$$

This is the scalar component of the velocity at (x, y) in the direction of the outward normal times the length of the segment. If the velocity is in meters per second, for example, the exit rate will be in meters per second times meters or square meters per second. The rates at which the fluid crosses the other three sides in the directions of their outward normals can be estimated in a similar way. All told, we have

Exit Rates:	Top:	$\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j}\, \Delta x = N(x, y + \Delta y)\Delta x$
	Bottom:	$\mathbf{F}(x, y) \cdot (-\mathbf{j})\, \Delta x = -N(x, y)\Delta x$
	Right:	$\mathbf{F}(x + \Delta x, y) \cdot \mathbf{i}\, \Delta y = M(x + \Delta x, y)\Delta y$
	Left:	$\mathbf{F}(x, y) \cdot (-\mathbf{i})\, \Delta y = -M(x, y)\Delta y.$

Combining opposite pairs gives

Top and bottom: $$(N(x, y + \Delta y) - N(x, y))\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

Right and left: $$(M(x + \Delta x, y) - M(x, y))\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y.$$

Adding these last two equations gives

$$\text{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y.$$

We now divide by $\Delta x \Delta y$ to estimate the total flux per unit area or flux density for the rectangle:

$$\frac{\text{Flux across rectangle boundary}}{\text{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right).$$

Finally, we let Δx and Δy approach zero to define what we call the *flux density* of $\mathbf{F}$ at the point (x, y). In mathematics, we call the flux density the *divergence* of $\mathbf{F}$. The symbol for it is div $\mathbf{F}$, pronounced "divergence of $\mathbf{F}$" or "div $\mathbf{F}$."

DEFINITION Divergence (Flux Density)

The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div}\,\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \qquad (1)$$

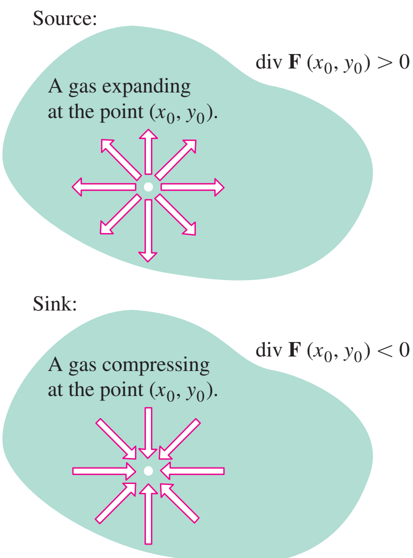

FIGURE 16.25 If a gas is expanding at a point (x_0, y_0), the lines of flow have positive divergence; if the gas is compressing, the divergence is negative.

Intuitively, if a gas is expanding at the point (x_0, y_0), the lines of flow would diverge there (hence the name) and, since the gas would be flowing out of a small rectangle about (x_0, y_0) the divergence of $\mathbf{F}$ at (x_0, y_0) would be positive. If the gas were compressing instead of expanding, the divergence would be negative (see Figure 16.25).

EXAMPLE 1 Finding Divergence

Find the divergence of $\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (xy - y^2)\mathbf{j}$.

Solution We use the formula in Equation (1):

$$\begin{aligned} \text{div } \mathbf{F} &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{\partial}{\partial x}(x^2 - y) + \frac{\partial}{\partial y}(xy - y^2) \\ &= 2x + x - 2y = 3x - 2y. \end{aligned}$$

Spin Around an Axis: The k-Component of Curl

The second idea we need for Green's Theorem has to do with measuring how a paddle wheel spins at a point in a fluid flowing in a plane region. This idea gives some sense of how the fluid is circulating around axes located at different points and perpendicular to the region. Physicists sometimes refer to this as the *circulation density* of a vector field $\mathbf{F}$ at a point. To obtain it, we return to the velocity field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

and the rectangle A. The rectangle is redrawn here as Figure 16.26.

The counterclockwise circulation of $\mathbf{F}$ around the boundary of A is the sum of flow rates along the sides. For the bottom edge, the flow rate is approximately

$$\mathbf{F}(x, y) \cdot \mathbf{i}\, \Delta x = M(x, y)\Delta x.$$

This is the scalar component of the velocity $\mathbf{F}(x, y)$ in the direction of the tangent vector $\mathbf{i}$ times the length of the segment. The rates of flow along the other sides in the counterclockwise direction are expressed in a similar way. In all, we have

$$\begin{aligned} &\text{Top:} && \mathbf{F}(x, y + \Delta y) \cdot (-\mathbf{i})\, \Delta x = -M(x, y + \Delta y)\Delta x \\ &\text{Bottom:} && \mathbf{F}(x, y) \cdot \mathbf{i}\, \Delta x = M(x, y)\Delta x \\ &\text{Right:} && \mathbf{F}(x + \Delta x, y) \cdot \mathbf{j}\, \Delta y = N(x + \Delta x, y)\Delta y \\ &\text{Left:} && \mathbf{F}(x, y) \cdot (-\mathbf{j})\, \Delta y = -N(x, y)\Delta y. \end{aligned}$$

We add opposite pairs to get

Top and bottom:

$$-(M(x, y + \Delta y) - M(x, y))\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

Right and left:

$$(N(x + \Delta x, y) - N(x, y))\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y.$$

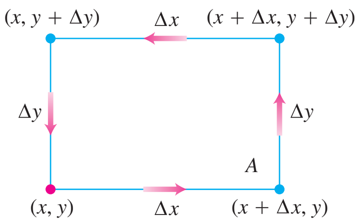

FIGURE 16.26 The rectangle for defining the curl (circulation density) of a vector field at a point (x, y).

Adding these last two equations and dividing by $\Delta x \Delta y$ gives an estimate of the circulation density for the rectangle:

$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

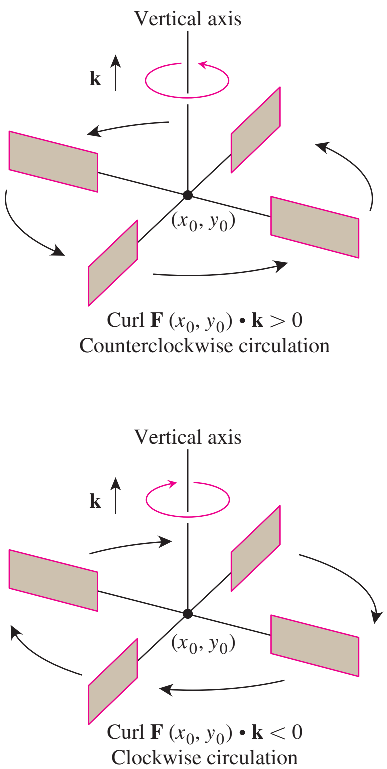

FIGURE 16.27 In the flow of an incompressible fluid over a plane region, the **k**-component of the curl measures the rate of the fluid's rotation at a point. The **k**-component of the curl is positive at points where the rotation is counterclockwise and negative where the rotation is clockwise.

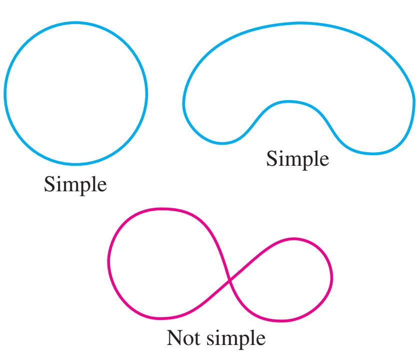

FIGURE 16.28 In proving Green's Theorem, we distinguish between two kinds of closed curves, simple and not simple. Simple curves do not cross themselves. A circle is simple but a figure 8 is not.

We let Δx and Δy approach zero to define what we call the *circulation density* of **F** at the point (x, y).

The positive orientation of the circulation density for the plane is the *counterclockwise* rotation around the vertical axis, looking downward on the xy-plane from the tip of the (vertical) unit vector **k** (Figure 16.27). The circulation value is actually the **k**-component of a more general circulation vector we define in Section 16.7, called the *curl* of the vector field **F**. For Green's Theorem, we need only this **k**-component.

DEFINITION k-Component of Curl (Circulation Density)

The k-component of the curl (circulation density) of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{2}$$

If water is moving about a region in the xy-plane in a thin layer, then the **k**-component of the circulation, or curl, at a point (x_0, y_0) gives a way to measure how fast and in what direction a small paddle wheel will spin if it is put into the water at (x_0, y_0) with its axis perpendicular to the plane, parallel to **k** (Figure 16.27).

EXAMPLE 2 Finding the **k**-Component of the Curl

Find the **k**-component of the curl for the vector field

$$\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (xy - y^2)\mathbf{j}.$$

Solution We use the formula in Equation (2):

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}(xy - y^2) - \frac{\partial}{\partial y}(x^2 - y) = y + 1.$$

■

Two Forms for Green's Theorem

In one form, Green's Theorem says that under suitable conditions the outward flux of a vector field across a simple closed curve in the plane (Figure 16.28) equals the double integral of the divergence of the field over the region enclosed by the curve. Recall the formulas for flux in Equations (3) and (4) in Section 16.2.

THEOREM 3 Green's Theorem (Flux-Divergence or Normal Form)

The outward flux of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ across a simple closed curve C equals the double integral of div **F** over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\, dy - N\, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy}_{\text{Divergence integral}} \tag{3}$$

In another form, Green's Theorem says that the counterclockwise circulation of a vector field around a simple closed curve is the double integral of the **k**-component of the curl of the field over the region enclosed by the curve. Recall the defining Equation (2) for circulation in Section 16.2.

THEOREM 4 Green's Theorem (Circulation-Curl or Tangential Form)

The counterclockwise circulation of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around a simple closed curve C in the plane equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy}_{\text{Curl integral}} \qquad (4)$$

The two forms of Green's Theorem are equivalent. Applying Equation (3) to the field $\mathbf{G}_1 = N\mathbf{i} - M\mathbf{j}$ gives Equation (4), and applying Equation (4) to $\mathbf{G}_2 = -N\mathbf{i} + M\mathbf{j}$ gives Equation (3).

Mathematical Assumptions

We need two kinds of assumptions for Green's Theorem to hold. First, we need conditions on M and N to ensure the existence of the integrals. The usual assumptions are that M, N, and their first partial derivatives are continuous at every point of some open region containing C and R. Second, we need geometric conditions on the curve C. It must be simple, closed, and made up of pieces along which we can integrate M and N. The usual assumptions are that C is piecewise smooth. The proof we give for Green's Theorem, however, assumes things about the shape of R as well. You can find proofs that are less restrictive in more advanced texts. First let's look at examples.

EXAMPLE 3 Supporting Green's Theorem

Verify both forms of Green's Theorem for the field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

$$C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \qquad 0 \le t \le 2\pi.$$

Solution We have

$$M = \cos t - \sin t, \qquad dx = d(\cos t) = -\sin t\, dt,$$

$$N = \cos t, \qquad dy = d(\sin t) = \cos t\, dt,$$

$$\frac{\partial M}{\partial x} = 1, \qquad \frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 1, \qquad \frac{\partial N}{\partial y} = 0.$$

The two sides of Equation (3) are

$$\oint_C M\,dy - N\,dx = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt)$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \pi$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1 + 0)\,dx\,dy$$

$$= \iint_R dx\,dy = \text{area inside the unit circle} = \pi.$$

The two sides of Equation (4) are

$$\oint_C M\,dx + N\,dy = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)$$

$$= \int_0^{2\pi} (-\sin t \cos t + 1)\,dt = 2\pi$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{dy}\right) dx\,dy = \iint_R (1 - (-1))\,dx\,dy = 2\iint_R dx\,dy = 2\pi.$$ ■

Using Green's Theorem to Evaluate Line Integrals

If we construct a closed curve C by piecing a number of different curves end to end, the process of evaluating a line integral over C can be lengthy because there are so many different integrals to evaluate. If C bounds a region R to which Green's Theorem applies, however, we can use Green's Theorem to change the line integral around C into one double integral over R.

EXAMPLE 4 Evaluating a Line Integral Using Green's Theorem

Evaluate the integral

$$\oint_C xy\,dy - y^2\,dx,$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Solution We can use either form of Green's Theorem to change the line integral into a double integral over the square.

1. *With the Normal Form Equation* (3): Taking $M = xy$, $N = y^2$, and C and R as the square's boundary and interior gives

$$\oint_C xy\,dy - y^2\,dx = \iint_R (y + 2y)\,dx\,dy = \int_0^1\int_0^1 3y\,dx\,dy$$

$$= \int_0^1 \left[3xy\right]_{x=0}^{x=1} dy = \int_0^1 3y\,dy = \frac{3}{2}y^2\Big]_0^1 = \frac{3}{2}.$$

2. *With the Tangential Form Equation* (4): Taking $M = -y^2$ and $N = xy$ gives the same result:

$$\oint_C -y^2\,dx + xy\,dy = \iint_R (y - (-2y))\,dx\,dy = \frac{3}{2}. \qquad \blacksquare$$

EXAMPLE 5 Finding Outward Flux

Calculate the outward flux of the field $\mathbf{F}(x, y) = x\mathbf{i} + y^2\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution Calculating the flux with a line integral would take four integrations, one for each side of the square. With Green's Theorem, we can change the line integral to one double integral. With $M = x$, $N = y^2$, C the square, and R the square's interior, we have

$$\begin{aligned}
\text{Flux} &= \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \qquad \text{Green's Theorem} \\
&= \int_{-1}^{1}\int_{-1}^{1} (1 + 2y)\,dx\,dy = \int_{-1}^{1} \Big[x + 2xy\Big]_{x=-1}^{x=1} dy \\
&= \int_{-1}^{1} (2 + 4y)\,dy = \Big[2y + 2y^2\Big]_{-1}^{1} = 4. \qquad \blacksquare
\end{aligned}$$

Proof of Green's Theorem for Special Regions

Let C be a smooth simple closed curve in the xy-plane with the property that lines parallel to the axes cut it in no more than two points. Let R be the region enclosed by C and suppose that M, N, and their first partial derivatives are continuous at every point of some open region containing C and R. We want to prove the circulation-curl form of Green's Theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \tag{5}$$

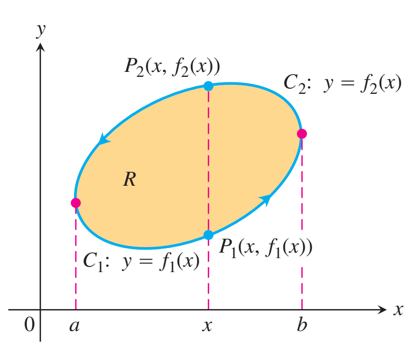

FIGURE 16.29 The boundary curve C is made up of C_1, the graph of $y = f_1(x)$, and C_2, the graph of $y = f_2(x)$.

Figure 16.29 shows C made up of two directed parts:

$$C_1:\quad y = f_1(x),\quad a \le x \le b, \qquad C_2:\quad y = f_2(x),\quad b \ge x \ge a.$$

For any x between a and b, we can integrate $\partial M/\partial y$ with respect to y from $y = f_1(x)$ to $y = f_2(x)$ and obtain

$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy = M(x, y)\Big]_{y=f_1(x)}^{y=f_2(x)} = M(x, f_2(x)) - M(x, f_1(x)).$$

We can then integrate this with respect to x from a to b:

$$\begin{aligned}\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\, dx &= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\, dx \\ &= -\int_b^a M(x, f_2(x))\, dx - \int_a^b M(x, f_1(x))\, dx \\ &= -\int_{C_2} M\, dx - \int_{C_1} M\, dx \\ &= -\oint_C M\, dx.\end{aligned}$$

Therefore

$$\oint_C M\, dx = \iint_R \left(-\frac{\partial M}{\partial y}\right) dx\, dy. \tag{6}$$

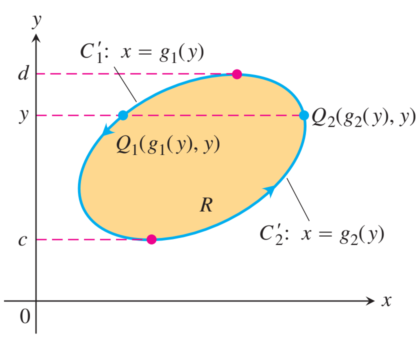

FIGURE 16.30 The boundary curve C is made up of C_1', the graph of $x = g_1(y)$, and C_2', the graph of $x = g_2(y)$.

Equation (6) is half the result we need for Equation (5). We derive the other half by integrating $\partial N/\partial x$ first with respect to x and then with respect to y, as suggested by Figure 16.30. This shows the curve C of Figure 16.29 decomposed into the two directed parts C_1': $x = g_1(y)$, $d \geq y \geq c$ and C_2': $x = g_2(y)$, $c \leq y \leq d$. The result of this double integration is

$$\oint_C N\, dy = \iint_R \frac{\partial N}{\partial x}\, dx\, dy. \tag{7}$$

Summing Equations (6) and (7) gives Equation (5). This concludes the proof. ■

Extending the Proof to Other Regions

The argument we just gave does not apply directly to the rectangular region in Figure 16.31 because the lines $x = a, x = b, y = c$, and $y = d$ meet the region's boundary in more than two points. If we divide the boundary C into four directed line segments, however,

$$\begin{aligned} &C_1: \quad y = c, \quad a \leq x \leq b, \qquad C_2: \quad x = b, \quad c \leq y \leq d \\ &C_3: \quad y = d, \quad b \geq x \geq a, \qquad C_4: \quad x = a, \quad d \geq y \geq c, \end{aligned}$$

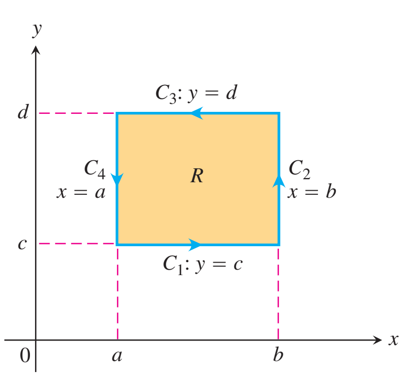

FIGURE 16.31 To prove Green's Theorem for a rectangle, we divide the boundary into four directed line segments.

we can modify the argument in the following way.

Proceeding as in the proof of Equation (7), we have

$$\begin{aligned}\int_c^d \int_a^b \frac{\partial N}{\partial x}\, dx\, dy &= \int_c^d (N(b, y) - N(a, y))\, dy \\ &= \int_c^d N(b, y)\, dy + \int_d^c N(a, y)\, dy \\ &= \int_{C_2} N\, dy + \int_{C_4} N\, dy. \end{aligned} \tag{8}$$

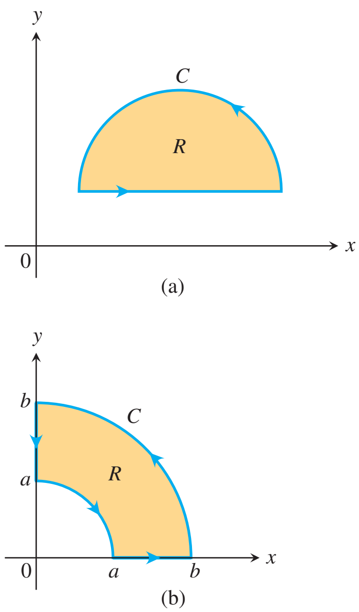

FIGURE 16.32 Other regions to which Green's Theorem applies.

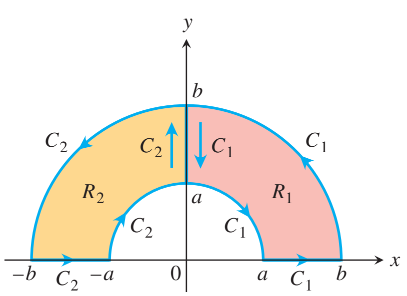

FIGURE 16.33 A region R that combines regions R_1 and R_2.

Because y is constant along C_1 and C_3, $\int_{C_1} N\,dy = \int_{C_3} N\,dy = 0$, so we can add $\int_{C_1} N\,dy = \int_{C_3} N\,dy$ to the right-hand side of Equation (8) without changing the equality. Doing so, we have

$$\int_c^d \int_a^b \frac{\partial N}{\partial x}\,dx\,dy = \oint_C N\,dy. \tag{9}$$

Similarly, we can show that

$$\int_a^b \int_c^d \frac{\partial M}{\partial y}\,dy\,dx = -\oint_C M\,dx. \tag{10}$$

Subtracting Equation (10) from Equation (9), we again arrive at

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Regions like those in Figure 16.32 can be handled with no greater difficulty. Equation (5) still applies. It also applies to the horseshoe-shaped region R shown in Figure 16.33, as we see by putting together the regions R_1 and R_2 and their boundaries. Green's Theorem applies to C_1, R_1 and to C_2, R_2, yielding

$$\int_{C_1} M\,dx + N\,dy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\int_{C_2} M\,dx + N\,dy = \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

When we add these two equations, the line integral along the y-axis from b to a for C_1 cancels the integral over the same segment but in the opposite direction for C_2. Hence,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy,$$

where C consists of the two segments of the x-axis from $-b$ to $-a$ and from a to b and of the two semicircles, and where R is the region inside C.

The device of adding line integrals over separate boundaries to build up an integral over a single boundary can be extended to any finite number of subregions. In Figure 16.34a let C_1 be the boundary, oriented counterclockwise, of the region R_1 in the first quadrant. Similarly, for the other three quadrants, C_i is the boundary of the region R_i, $i = 2, 3, 4$. By Green's Theorem,

$$\oint_{C_i} M\,dx + N\,dy = \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \tag{11}$$

We sum Equation (11) over $i = 1, 2, 3, 4$, and get (Figure 16.34b):

$$\oint_{r=b} (M\,dx + N\,dy) + \oint_{r=a} (M\,dx + N\,dy) = \iint_{\cup R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \tag{12}$$

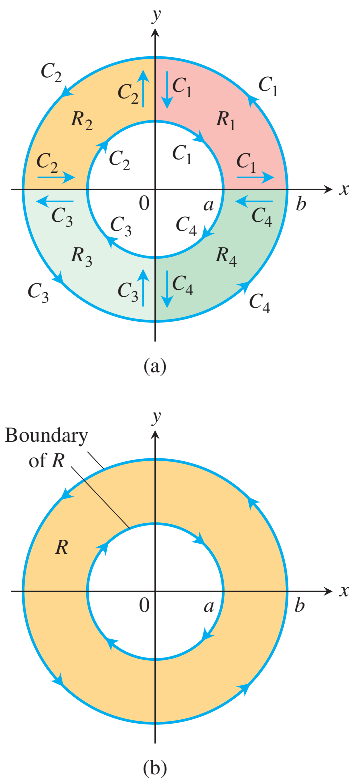

FIGURE 16.34 The annular region R combines four smaller regions. In polar coordinates, $r = a$ for the inner circle, $r = b$ for the outer circle, and $a \leq r \leq b$ for the region itself.

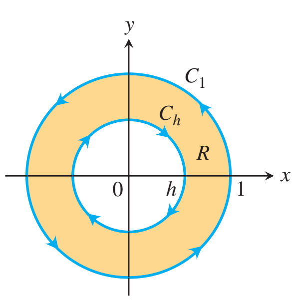

FIGURE 16.35 Green's Theorem may be applied to the annular region R by integrating along the boundaries as shown (Example 6).

Equation (12) says that the double integral of $(\partial N/\partial x) - (\partial M/\partial y)$ over the annular ring R equals the line integral of $M\,dx + N\,dy$ over the complete boundary of R in the direction that keeps R on our left as we progress (Figure 16.34b).

EXAMPLE 6 Verifying Green's Theorem for an Annular Ring

Verify the circulation form of Green's Theorem (Equation 4) on the annular ring R: $h^2 \leq x^2 + y^2 \leq 1$, $0 < h < 1$ (Figure 16.35), if

$$M = \frac{-y}{x^2 + y^2}, \qquad N = \frac{x}{x^2 + y^2}.$$

Solution The boundary of R consists of the circle

$$C_1: \quad x = \cos t, \qquad y = \sin t, \qquad 0 \leq t \leq 2\pi,$$

traversed counterclockwise as t increases, and the circle

$$C_h: \quad x = h\cos\theta, \qquad y = -h\sin\theta, \qquad 0 \leq \theta \leq 2\pi,$$

traversed clockwise as θ increases. The functions M and N and their partial derivatives are continuous throughout R. Moreover,

$$\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)(-1) + y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x},$$

so

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0.$$

The integral of $M\,dx + N\,dy$ over the boundary of R is

$$\int_C M\,dx + N\,dy = \oint_{C_1} \frac{x\,dy - y\,dx}{x^2 + y^2} + \oint_{C_h} \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt - \int_0^{2\pi} \frac{h^2(\cos^2\theta + \sin^2\theta)}{h^2}\,d\theta$$

$$= 2\pi - 2\pi = 0.$$ ■

The functions M and N in Example 6 are discontinuous at $(0, 0)$, so we cannot apply Green's Theorem to the circle C_1 and the region inside it. We must exclude the origin. We do so by excluding the points interior to C_h.

We could replace the circle C_1 in Example 6 by an ellipse or any other simple closed curve K surrounding C_h (Figure 16.36). The result would still be

$$\oint_K (M\,dx + N\,dy) + \oint_{C_h} (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = 0,$$

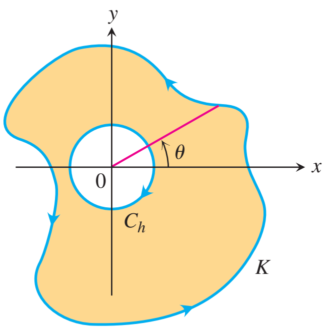

FIGURE 16.36 The region bounded by the circle C_h and the curve K.

which leads to the conclusion that

$$\oint_K (M\,dx + N\,dy) = 2\pi$$

for any such curve K. We can explain this result by changing to polar coordinates. With

$$x = r\cos\theta, \qquad y = r\sin\theta,$$
$$dx = -r\sin\theta\,d\theta + \cos\theta\,dr, \qquad dy = r\cos\theta\,d\theta + \sin\theta\,dr,$$

we have

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{r^2(\cos^2\theta + \sin^2\theta)\,d\theta}{r^2} = d\theta,$$

and θ increases by 2π as we traverse K once counterclockwise.