16.5 Surface Area and Surface Integrals

We know how to integrate a function over a flat region in a plane, but what if the function is defined over a curved surface? To evaluate one of these so-called surface integrals, we rewrite it as a double integral over a region in a coordinate plane beneath the surface (Figure 16.38). Surface integrals are used to compute quantities such as the flow of liquid across a membrane or the upward force on a falling parachute.

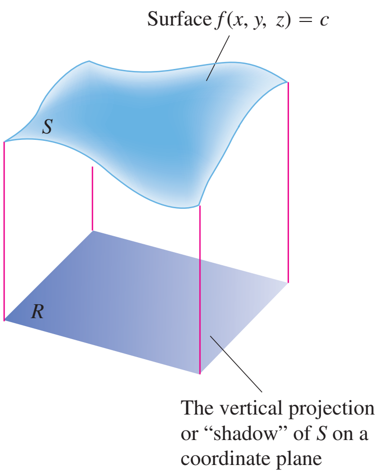

FIGURE 16.38 As we soon see, the integral of a function $g(x, y, z)$ over a surface S in space can be calculated by evaluating a related double integral over the vertical projection or "shadow" of S on a coordinate plane.

Surface Area

Figure 16.39 shows a surface S lying above its "shadow" region R in a plane beneath it. The surface is defined by the equation $f(x, y, z) = c$. If the surface is **smooth** (∇f is continuous and never vanishes on S), we can define and calculate its area as a double integral over R. We assume that this projection of the surface onto its shadow R is one-to-one. That is, each point in R corresponds to exactly one point (x, y, z) satisfying $f(x, y, z) = c$.

The first step in defining the area of S is to partition the region R into small rectangles ΔA_k of the kind we would use if we were defining an integral over R. Directly above each ΔA_k lies a patch of surface $\Delta\sigma_k$ that we may approximate by a parallelogram ΔP_k in the tangent plane to S at a point $T_k(x_k, y_k, z_k)$ in $\Delta\sigma_k$. This parallelogram in the tangent plane projects directly onto ΔA_k. To be specific, we choose the point $T_k(x_k, y_k, z_k)$ lying directly above the back corner C_k of ΔA_k, as shown in Figure 16.39. If the tangent plane is parallel to R, then ΔP_k will be congruent to ΔA_k. Otherwise, it will be a parallelogram whose area is somewhat larger than the area of ΔA_k.

Figure 16.40 gives a magnified view of $\Delta\sigma_k$ and ΔP_k, showing the gradient vector $\nabla f(x_k, y_k, z_k)$ at T_k and a unit vector $\mathbf{p}$ that is normal to R. The figure also shows the angle γ_k between ∇f and $\mathbf{p}$. The other vectors in the picture, $\mathbf{u}_k$ and $\mathbf{v}_k$, lie along the edges of the patch ΔP_k in the tangent plane. Thus, both $\mathbf{u}_k \times \mathbf{v}_k$ and ∇f are normal to the tangent plane.

We now need to know from advanced vector geometry that $|(\mathbf{u}_k \times \mathbf{v}_k)\cdot\mathbf{p}|$ is the area of the projection of the parallelogram determined by $\mathbf{u}_k$ and $\mathbf{v}_k$ onto any plane whose normal is $\mathbf{p}$. (A proof is given in Appendix 8.) In our case, this translates into the statement

$$|(\mathbf{u}_k \times \mathbf{v}_k)\cdot\mathbf{p}| = \Delta A_k.$$

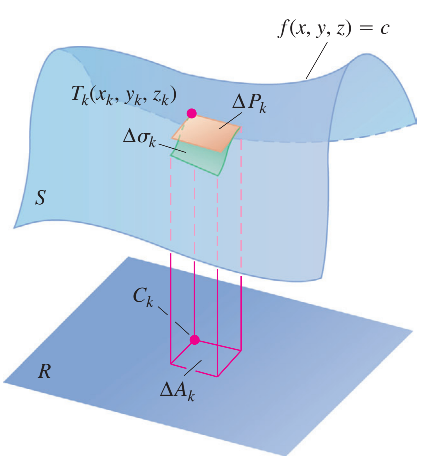

FIGURE 16.39 A surface S and its vertical projection onto a plane beneath it. You can think of R as the shadow of S on the plane. The tangent plane ΔP_k approximates the surface patch $\Delta\sigma_k$ above ΔA_k.

To simplify the notation in the derivation that follows, we are now denoting the *area* of the small rectangular region by ΔA_k as well. Likewise, ΔP_k will also denote the area of the portion of the tangent plane directly above this small region.

Now, $|\mathbf{u}_k \times \mathbf{v}_k|$ itself is the area ΔP_k (standard fact about cross products) so this last equation becomes

$$\underbrace{|\mathbf{u}_k \times \mathbf{v}_k|}_{\Delta P_k}\;\underbrace{|\mathbf{p}|}_{1}\;\underbrace{|\cos(\text{angle between } \mathbf{u}_k \times \mathbf{v}_k \text{ and } \mathbf{p})|}_{\substack{\text{Same as } |\cos\gamma_k| \text{ because } \nabla f \text{ and } \mathbf{u}_k\times\mathbf{v}_k \\ \text{are both normal to the tangent plane}}} = \Delta A_k$$

or

$$\Delta P_k |\cos\gamma_k| = \Delta A_k$$

or

$$\Delta P_k = \frac{\Delta A_k}{|\cos\gamma_k|},$$

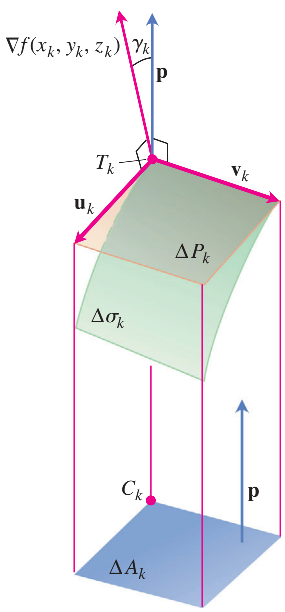

FIGURE 16.40 Magnified view from the preceding figure. The vector $\mathbf{u}_k \times \mathbf{v}_k$ (not shown) is parallel to the vector ∇f because both vectors are normal to the plane of ΔP_k.

provided $\cos\gamma_k \neq 0$. We will have $\cos\gamma_k \neq 0$ as long as ∇f is not parallel to the ground plane and $\nabla f \cdot \mathbf{p} \neq 0$.

Since the patches ΔP_k approximate the surface patches $\Delta\sigma_k$ that fit together to make S, the sum

$$\sum \Delta P_k = \sum \frac{\Delta A_k}{|\cos\gamma_k|} \tag{1}$$

looks like an approximation of what we might like to call the surface area of S. It also looks as if the approximation would improve if we refined the partition of R. In fact, the sums on the right-hand side of Equation (1) are approximating sums for the double integral

$$\iint_R \frac{1}{|\cos\gamma|}\, dA. \tag{2}$$

We therefore define the **area** of S to be the value of this integral whenever it exists. For any surface $f(x, y, z) = c$, we have $|\nabla f \cdot \mathbf{p}| = |\nabla f||\mathbf{p}||\cos\gamma|$, so

$$\frac{1}{|\cos\gamma|} = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}.$$

This combines with Equation (2) to give a practical formula for surface area.

Formula for Surface Area

The area of the surface $f(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA, \tag{3}$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$.

Thus, the area is the double integral over R of the magnitude of ∇f divided by the magnitude of the scalar component of ∇f normal to R.

We reached Equation (3) under the assumption that $\nabla f \cdot \mathbf{p} \neq 0$ throughout R and that ∇f is continuous. Whenever the integral exists, however, we define its value to be the area of the portion of the surface $f(x, y, z) = c$ that lies over R. (Recall that the projection is assumed to be one-to-one.)

In the exercises (see Equation 11), we show how Equation (3) simplifies if the surface is defined by $z = f(x, y)$.

EXAMPLE 1 Finding Surface Area

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

Solution We sketch the surface S and the region R below it in the xy-plane (Figure 16.41). The surface S is part of the level surface $f(x, y, z) = x^2 + y^2 - z = 0$, and R is the disk $x^2 + y^2 \leq 4$ in the xy-plane. To get a unit vector normal to the plane of R, we can take $\mathbf{p} = \mathbf{k}$.

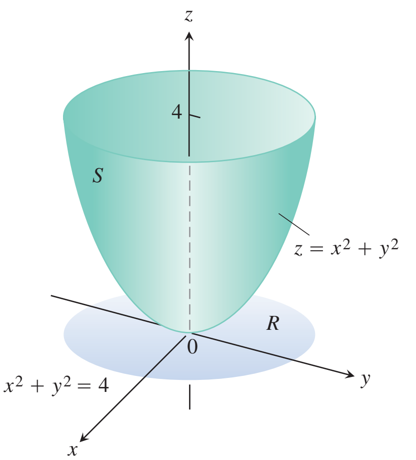

FIGURE 16.41 The area of this parabolic surface is calculated in Example 1.

At any point (x, y, z) on the surface, we have

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 - z \\ \nabla f &= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \\ |\nabla f| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla f \cdot \mathbf{p}| &= |\nabla f \cdot \mathbf{k}| = |-1| = 1. \end{aligned}$$

In the region R, $dA = dx\,dy$. Therefore,

$$\begin{aligned} \text{Surface area} &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA \qquad \text{Equation (3)} \\ &= \iint_{x^2+y^2 \le 4} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy \\ &= \int_0^{2\pi}\int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta \qquad \text{Polar coordinates} \\ &= \int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{1}{12}(17^{3/2} - 1)\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right). \end{aligned}$$

■

EXAMPLE 2 Finding Surface Area

Find the area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2$, $z \ge 0$, by the cylinder $x^2 + y^2 = 1$ (Figure 16.42).

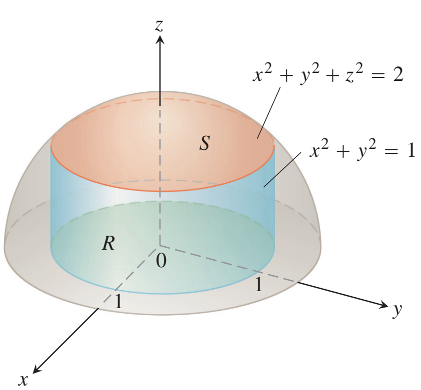

FIGURE 16.42 The cap cut from the hemisphere by the cylinder projects vertically onto the disk R: $x^2 + y^2 \le 1$ in the xy-plane (Example 2).

Solution The cap S is part of the level surface $f(x, y, z) = x^2 + y^2 + z^2 = 2$. It projects one-to-one onto the disk R: $x^2 + y^2 \le 1$ in the xy-plane. The unit vector $\mathbf{p} = \mathbf{k}$ is normal to the plane of R.

At any point on the surface,

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 + z^2 \\ \nabla f &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ |\nabla f| &= 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2} \qquad \text{Because } x^2 + y^2 + z^2 = 2 \text{ at points of } S \\ |\nabla f \cdot \mathbf{p}| &= |\nabla f \cdot \mathbf{k}| = |2z| = 2z. \end{aligned}$$

Therefore,

$$\text{Surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \frac{2\sqrt{2}}{2z}\, dA = \sqrt{2}\iint_R \frac{dA}{z}. \tag{4}$$

What do we do about the z?

Since z is the z-coordinate of a point on the sphere, we can express it in terms of x and y as

$$z = \sqrt{2 - x^2 - y^2}.$$

We continue the work of Equation (4) with this substitution:

$$\text{Surface area} = \sqrt{2}\iint_R \frac{dA}{z} = \sqrt{2}\iint_{x^2+y^2\le 1} \frac{dA}{\sqrt{2-x^2-y^2}}$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{\sqrt{2-r^2}} \qquad \text{Polar coordinates}$$

$$= \sqrt{2}\int_0^{2\pi}\left[-(2-r^2)^{1/2}\right]_{r=0}^{r=1} d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\sqrt{2}-1\right)d\theta = 2\pi\left(2-\sqrt{2}\right).$$

Surface Integrals

We now show how to integrate a function over a surface, using the ideas just developed for calculating surface area.

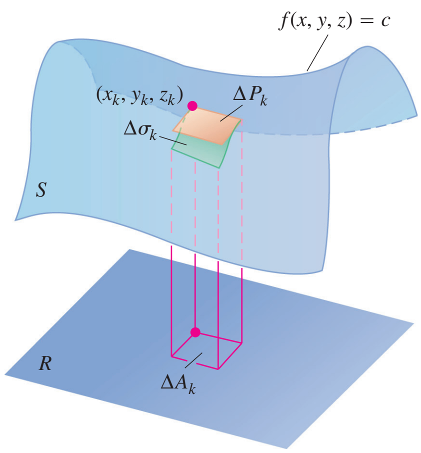

FIGURE 16.43 If we know how an electrical charge $g(x, y, z)$ is distributed over a surface, we can find the total charge with a suitably modified surface integral.

Suppose, for example, that we have an electrical charge distributed over a surface $f(x, y, z) = c$ like the one shown in Figure 16.43 and that the function $g(x, y, z)$ gives the charge per unit area (charge density) at each point on S. Then we may calculate the total charge on S as an integral in the following way.

We partition the shadow region R on the ground plane beneath the surface into small rectangles of the kind we would use if we were defining the surface area of S. Then directly above each ΔA_k lies a patch of surface $\Delta\sigma_k$ that we approximate with a parallelogram-shaped portion of tangent plane, ΔP_k. (See Figure 16.43.)

Up to this point the construction proceeds as in the definition of surface area, but now we take an additional step: We evaluate g at (x_k, y_k, z_k) and approximate the total charge on the surface path $\Delta\sigma_k$ by the product $g(x_k, y_k, z_k)\,\Delta P_k$. The rationale is that when the partition of R is sufficiently fine, the value of g throughout $\Delta\sigma_k$ is nearly constant and ΔP_k is nearly the same as $\Delta\sigma_k$. The total charge over S is then approximated by the sum

$$\text{Total charge} \approx \sum g(x_k, y_k, z_k)\,\Delta P_k = \sum g(x_k, y_k, z_k)\frac{\Delta A_k}{|\cos\gamma_k|}.$$

If f, the function defining the surface S, and its first partial derivatives are continuous, and if g is continuous over S, then the sums on the right-hand side of the last equation approach the limit

$$\iint_R g(x, y, z)\frac{dA}{|\cos\gamma|} = \iint_R g(x, y, z)\frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA \qquad (5)$$

as the partition of R is refined in the usual way. This limit is called the integral of g over the surface S and is calculated as a double integral over R. The value of the integral is the total charge on the surface S.

As you might expect, the formula in Equation (5) defines the integral of *any* function g over the surface S as long as the integral exists.

DEFINITION Surface Integral

If R is the shadow region of a surface S defined by the equation $f(x, y, z) = c$, and g is a continuous function defined at the points of S, then the **integral of g over S** is the integral

$$\iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} \, dA, \tag{6}$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$. The integral itself is called a **surface integral**.

The integral in Equation (6) takes on different meanings in different applications. If g has the constant value 1, the integral gives the area of S. If g gives the mass density of a thin shell of material modeled by S, the integral gives the mass of the shell.

We can abbreviate the integral in Equation (6) by writing $d\sigma$ for $(|\nabla f| / |\nabla f \cdot \mathbf{p}|) \, dA$.

The Surface Area Differential and the Differential Form for Surface Integrals

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} \, dA \qquad \iint_S g \, d\sigma \tag{7}$$

Surface area differential — Differential formula for surface integrals

Surface integrals behave like other double integrals, the integral of the sum of two functions being the sum of their integrals and so on. The domain Additivity Property takes the form

$$\iint_S g \, d\sigma = \iint_{S_1} g \, d\sigma + \iint_{S_2} g \, d\sigma + \cdots + \iint_{S_n} g \, d\sigma.$$

The idea is that if S is partitioned by smooth curves into a finite number of nonoverlapping smooth patches (i.e., if S is **piecewise smooth**), then the integral over S is the sum of the integrals over the patches. Thus, the integral of a function over the surface of a cube is the sum of the integrals over the faces of the cube. We integrate over a turtle shell of welded plates by integrating one plate at a time and adding the results.

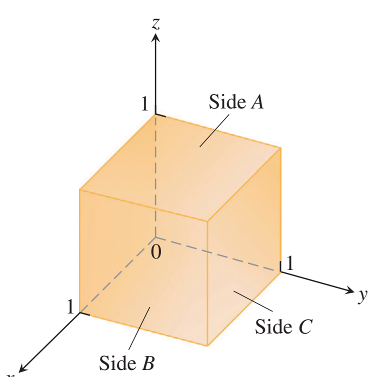

FIGURE 16.44 The cube in Example 3.

EXAMPLE 3 Integrating Over a Surface

Integrate $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ (Figure 16.44).

Solution We integrate xyz over each of the six sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

$$\iint_{\substack{\text{Cube} \\ \text{surface}}} xyz \, d\sigma = \iint_{\text{Side } A} xyz \, d\sigma + \iint_{\text{Side } B} xyz \, d\sigma + \iint_{\text{Side } C} xyz \, d\sigma.$$

Side A is the surface $f(x, y, z) = z = 1$ over the square region R_{xy}: $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane. For this surface and region,

$$\mathbf{p} = \mathbf{k}, \quad \nabla f = \mathbf{k}, \quad |\nabla f| = 1, \quad |\nabla f \cdot \mathbf{p}| = |\mathbf{k} \cdot \mathbf{k}| = 1$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \frac{1}{1}\, dx\, dy = dx\, dy$$

$$xyz = xy(1) = xy$$

and

$$\iint_{\text{Side } A} xyz\, d\sigma = \iint_{R_{xy}} xy\, dx\, dy = \int_0^1 \int_0^1 xy\, dx\, dy = \int_0^1 \frac{y}{2}\, dy = \frac{1}{4}.$$

Symmetry tells us that the integrals of xyz over sides B and C are also $1/4$. Hence,

$$\iint_{\substack{\text{Cube}\\ \text{surface}}} xyz\, d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

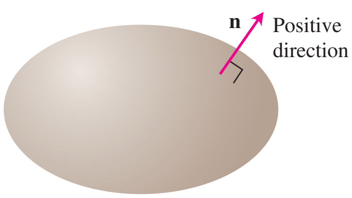

FIGURE 16.45 Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.

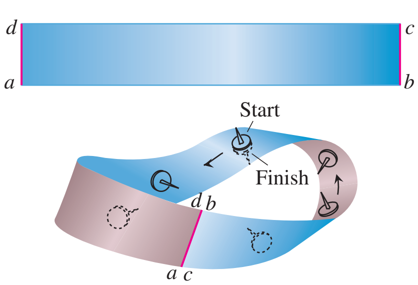

FIGURE 16.46 To make a Möbius band, take a rectangular strip of paper *abcd*, give the end *bc* a single twist, and paste the ends of the strip together to match *a* with *c* and *b* with *d*. The Möbius band is a nonorientable or one-sided surface.

Orientation

We call a smooth surface S **orientable** or **two-sided** if it is possible to define a field $\mathbf{n}$ of unit normal vectors on S that varies continuously with position. Any patch or subportion of an orientable surface is orientable. Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable. By convention, we choose $\mathbf{n}$ on a closed surface to point outward.

Once $\mathbf{n}$ has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector $\mathbf{n}$ at any point is called the **positive direction** at that point (Figure 16.45).

The Möbius band in Figure 16.46 is not orientable. No matter where you start to construct a continuous-unit normal field (shown as the shaft of a thumbtack in the figure), moving the vector continuously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out. The vector at that point cannot point both ways and yet it must if the field is to be continuous. We conclude that no such field exists.

Surface Integral for Flux

Suppose that $\mathbf{F}$ is a continuous vector field defined over an oriented surface S and that $\mathbf{n}$ is the chosen unit normal field on the surface. We call the integral of $\mathbf{F} \cdot \mathbf{n}$ over S the flux of $\mathbf{F}$ across S in the positive direction. Thus, the flux is the integral over S of the scalar component of $\mathbf{F}$ in the direction of $\mathbf{n}$.

DEFINITION Flux

The **flux** of a three-dimensional vector field $\mathbf{F}$ across an oriented surface S in the direction of $\mathbf{n}$ is

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma. \qquad (8)$$

The definition is analogous to the flux of a two-dimensional field **F** across a plane curve C. In the plane (Section 16.2), the flux is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

the integral of the scalar component of **F** normal to the curve.

If **F** is the velocity field of a three-dimensional fluid flow, the flux of **F** across S is the net rate at which fluid is crossing S in the chosen positive direction. We discuss such flows in more detail in Section 16.7.

If **S** is part of a level surface $g(x, y, z) = c$, then **n** may be taken to be one of the two fields

$$\mathbf{n} = \pm\frac{\nabla g}{|\nabla g|}, \tag{9}$$

depending on which one gives the preferred direction. The corresponding flux is

$$\begin{aligned} \text{Flux} &= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \\ &= \iint_R \left(\mathbf{F} \cdot \frac{\pm\nabla g}{|\nabla g|}\right) \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|}\, dA \qquad \text{Equations (9) and (7)} \qquad (8) \\ &= \iint_R \mathbf{F} \cdot \frac{\pm\nabla g}{|\nabla g \cdot \mathbf{p}|}\, dA. \qquad (10) \end{aligned}$$

EXAMPLE 4 Finding Flux

Find the flux of $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1$, $z \ge 0$, by the planes $x = 0$ and $x = 1$.

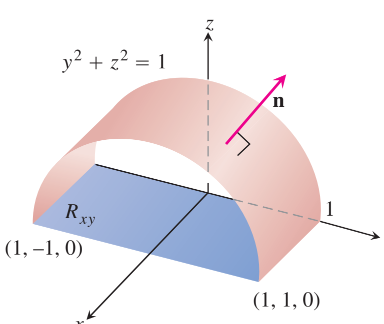

FIGURE 16.47 Calculating the flux of a vector field outward through this surface. The area of the shadow region R_{xy} is 2 (Example 4).

Solution The outward normal field on S (Figure 16.47) may be calculated from the gradient of $g(x, y, z) = y^2 + z^2$ to be

$$\mathbf{n} = +\frac{\nabla g}{|\nabla g|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{1}} = y\mathbf{j} + z\mathbf{k}.$$

With $\mathbf{p} = \mathbf{k}$, we also have

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{k}|}\, dA = \frac{2}{|2z|}\, dA = \frac{1}{z}\, dA.$$

We can drop the absolute value bars because $z \ge 0$ on S.

The value of $\mathbf{F} \cdot \mathbf{n}$ on the surface is

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= (yz\mathbf{j} + z^2\mathbf{k}) \cdot (y\mathbf{j} + z\mathbf{k}) \\ &= y^2 z + z^3 = z(y^2 + z^2) \\ &= z. \qquad y^2 + z^2 = 1 \text{ on } S \end{aligned}$$

Therefore, the flux of **F** outward through S is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S (z)\left(\frac{1}{z}\, dA\right) = \iint_{R_{xy}} dA = \text{area}(R_{xy}) = 2.$$ ■

Moments and Masses of Thin Shells

Thin shells of material like bowls, metal drums, and domes are modeled with surfaces. Their moments and masses are calculated with the formulas in Table 16.3.

TABLE 16.3 Mass and moment formulas for very thin shells

Mass: $M = \iint_S \delta(x, y, z)\, d\sigma \qquad (\delta(x, y, z) = \text{density at } (x, y, z), \text{ mass per unit area})$

First moments about the coordinate planes:

$$M_{yz} = \iint_S x\,\delta\, d\sigma, \qquad M_{xz} = \iint_S y\,\delta\, d\sigma, \qquad M_{xy} = \iint_S z\,\delta\, d\sigma$$

Coordinates of center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about coordinate axes:

$$I_x = \iint_S (y^2 + z^2)\,\delta\, d\sigma, \qquad I_y = \iint_S (x^2 + z^2)\,\delta\, d\sigma,$$

$$I_z = \iint_S (x^2 + y^2)\,\delta\, d\sigma, \qquad I_L = \iint_S r^2\delta\, d\sigma,$$

$$r(x, y, z) = \text{distance from point } (x, y, z) \text{ to line } L$$

Radius of gyration about a line L: $R_L = \sqrt{I_L/M}$

EXAMPLE 5 Finding Center of Mass

Find the center of mass of a thin hemispherical shell of radius a and constant density δ.

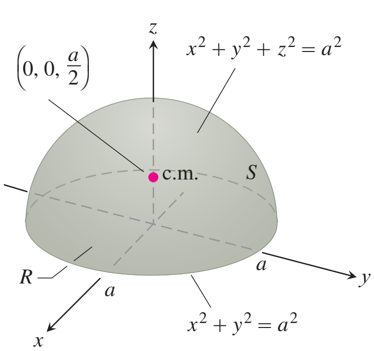

FIGURE 16.48 The center of mass of a thin hemispherical shell of constant density lies on the axis of symmetry halfway from the base to the top (Example 5).

Solution We model the shell with the hemisphere

$$f(x, y, z) = x^2 + y^2 + z^2 = a^2, \qquad z \geq 0$$

(Figure 16.48). The symmetry of the surface about the z-axis tells us that $\bar{x} = \bar{y} = 0$. It remains only to find $\bar{z}$ from the formula $\bar{z} = M_{xy}/M$.

The mass of the shell is

$$M = \iint_S \delta\, d\sigma = \delta \iint_S d\sigma = (\delta)(\text{area of } S) = 2\pi a^2 \delta.$$

To evaluate the integral for M_{xy}, we take $\mathbf{p} = \mathbf{k}$ and calculate

$$|\nabla f| = |2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}| = 2\sqrt{x^2 + y^2 + z^2} = 2a$$

$$|\nabla f \cdot \mathbf{p}| = |\nabla f \cdot \mathbf{k}| = |2z| = 2z$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \frac{a}{z}\, dA.$$

Then

$$M_{xy} = \iint_S z\delta\, d\sigma = \delta \iint_R z \frac{a}{z}\, dA = \delta a \iint_R dA = \delta a(\pi a^2) = \delta \pi a^3$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi a^3 \delta}{2\pi a^2 \delta} = \frac{a}{2}.$$

The shell's center of mass is the point $(0, 0, a/2)$. ■