

EXERCISES 16.5

Surface Area

- Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.
- Find the area of the band cut from the paraboloid $x^2 + y^2 - z = 0$ by the planes $z = 2$ and $z = 6$.
- Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.
- Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the xy -plane.
- Find the area of the surface $x^2 - 2y - 2z = 0$ that lies above the triangle bounded by the lines $x = 2$, $y = 0$, and $y = 3x$ in the xy -plane.
- Find the area of the cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.
- Find the area of the ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.
- Find the area of the upper portion of the cylinder $x^2 + z^2 = 1$ that lies between the planes $x = \pm 1/2$ and $y = \pm 1/2$.
- Find the area of the portion of the paraboloid $x = 4 - y^2 - z^2$ that lies above the ring $1 \leq y^2 + z^2 \leq 4$ in the yz -plane.
- Find the area of the surface cut from the paraboloid $x^2 + y + z^2 = 2$ by the plane $y = 0$.
- Find the area of the surface $x^2 - 2 \ln x + \sqrt{15}y - z = 0$ above the square $R: 1 \leq x \leq 2, 0 \leq y \leq 1$, in the xy -plane.
- Find the area of the surface $2x^{3/2} + 2y^{3/2} - 3z = 0$ above the square $R: 0 \leq x \leq 1, 0 \leq y \leq 1$, in the xy -plane.

Surface Integrals

- Integrate $g(x, y, z) = x + y + z$ over the surface of the cube cut from the first octant by the planes $x = a, y = a, z = a$.

- Integrate $g(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.
- Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid cut from the first octant by the planes $x = a, y = b$, and $z = c$.
- Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid bounded by the planes $x = \pm a, y = \pm b$, and $z = \pm c$.
- Integrate $g(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.
- Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0, x = 1$, and $z = 0$.

Flux Across a Surface

In Exercises 19 and 20, find the flux of the field \mathbf{F} across the portion of the given surface in the specified direction.

- $\mathbf{F}(x, y, z) = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

S : rectangular surface $z = 0, 0 \leq x \leq 2, 0 \leq y \leq 3$, direction \mathbf{k}

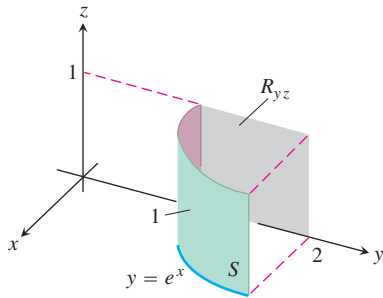
- $\mathbf{F}(x, y, z) = yx^2\mathbf{i} - 2\mathbf{j} + xz\mathbf{k}$

S : rectangular surface $y = 0, -1 \leq x \leq 2, 2 \leq z \leq 7$, direction $-\mathbf{j}$

In Exercises 21–26, find the flux of the field \mathbf{F} across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

- $\mathbf{F}(x, y, z) = z\mathbf{k}$
- $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$
- $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$
- $\mathbf{F}(x, y, z) = zx\mathbf{i} + zy\mathbf{j} + z^2\mathbf{k}$
- $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- $\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

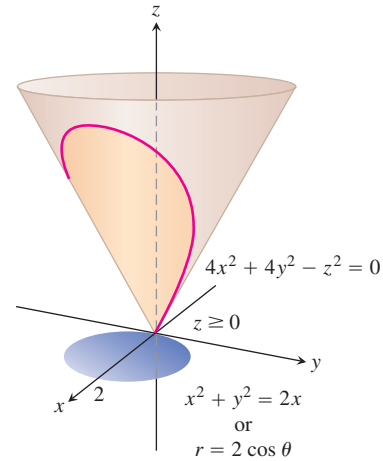
27. Find the flux of the field $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.
28. Find the flux of the field $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (away from the z -axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.
29. Let S be the portion of the cylinder $y = e^x$ in the first octant that projects parallel to the x -axis onto the rectangle R_{yz} : $1 \leq y \leq 2$, $0 \leq z \leq 1$ in the yz -plane (see the accompanying figure). Let \mathbf{n} be the unit vector normal to S that points away from the yz -plane. Find the flux of the field $\mathbf{F}(x, y, z) = -2\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$ across S in the direction of \mathbf{n} .



30. Let S be the portion of the cylinder $y = \ln x$ in the first octant whose projection parallel to the y -axis onto the xz -plane is the rectangle R_{xz} : $1 \leq x \leq e$, $0 \leq z \leq 1$. Let \mathbf{n} be the unit vector normal to S that points away from the xz -plane. Find the flux of $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ through S in the direction of \mathbf{n} .
31. Find the outward flux of the field $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the surface of the cube cut from the first octant by the planes $x = a$, $y = a$, $z = a$.
32. Find the outward flux of the field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \leq 25$ by the plane $z = 3$.

Moments and Masses

33. **Centroid** Find the centroid of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.
34. **Centroid** Find the centroid of the surface cut from the cylinder $y^2 + z^2 = 9$, $z \geq 0$, by the planes $x = 0$ and $x = 3$ (resembles the surface in Example 4).
35. **Thin shell of constant density** Find the center of mass and the moment of inertia and radius of gyration about the z -axis of a thin shell of constant density δ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.
36. **Conical surface of constant density** Find the moment of inertia about the z -axis of a thin shell of constant density δ cut from the cone $4x^2 + 4y^2 - z^2 = 0$, $z \geq 0$, by the circular cylinder $x^2 + y^2 = 2x$ (see the accompanying figure).



37. Spherical shells

- a. Find the moment of inertia about a diameter of a thin spherical shell of radius a and constant density δ . (Work with a hemispherical shell and double the result.)
- b. Use the Parallel Axis Theorem (Exercises 15.5) and the result in part (a) to find the moment of inertia about a line tangent to the shell.

38. a. **Cones with and without ice cream** Find the centroid of the lateral surface of a solid cone of base radius a and height h (cone surface minus the base).
- b. Use Pappus's formula (Exercises 15.5) and the result in part (a) to find the centroid of the complete surface of a solid cone (side plus base).
- c. A cone of radius a and height h is joined to a hemisphere of radius a to make a surface S that resembles an ice cream cone. Use Pappus's formula and the results in part (a) and Example 5 to find the centroid of S . How high does the cone have to be to place the centroid in the plane shared by the bases of the hemisphere and cone?

Special Formulas for Surface Area

If S is the surface defined by a function $z = f(x, y)$ that has continuous first partial derivatives throughout a region R_{xy} in the xy -plane (Figure 16.49), then S is also the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = f(x, y) - z$. Taking the unit normal to R_{xy} to be $\mathbf{p} = \mathbf{k}$ then gives

$$|\nabla F| = |f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}| = \sqrt{f_x^2 + f_y^2 + 1}$$

$$|\nabla F \cdot \mathbf{p}| = |(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}) \cdot \mathbf{k}| = |-1| = 1$$

and

$$\iint_{R_{xy}} \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA = \iint_{R_{xy}} \sqrt{f_x^2 + f_y^2 + 1} dx dy, \quad (11)$$

Similarly, the area of a smooth surface $x = f(y, z)$ over a region R_{yz} in the yz -plane is

$$A = \iint_{R_{yz}} \sqrt{f_y^2 + f_z^2 + 1} \, dy \, dz, \quad (12)$$

and the area of a smooth $y = f(x, z)$ over a region R_{xz} in the xz -plane is

$$A = \iint_{R_{xz}} \sqrt{f_x^2 + f_z^2 + 1} \, dx \, dz. \quad (13)$$

Use Equations (11)–(13) to find the area of the surfaces in Exercises 39–44.

39. The surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 3$
40. The surface cut from the “nose” of the paraboloid $x = 1 - y^2 - z^2$ by the yz -plane
41. The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies over the region between the circle $x^2 + y^2 = 1$ and the ellipse $9x^2 + 4y^2 = 36$ in the xy -plane. (*Hint:* Use formulas from geometry to find the area of the region.)
42. The triangle cut from the plane $2x + 6y + 3z = 6$ by the bounding planes of the first octant. Calculate the area three ways, once with each area formula

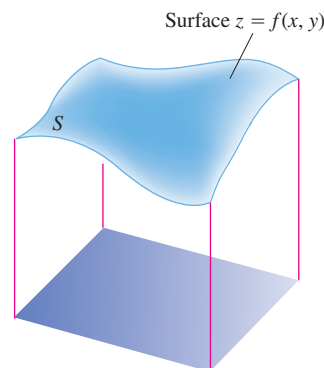


FIGURE 16.49 For a surface $z = f(x, y)$, the surface area formula in Equation (3) takes the form

$$A = \iint_{R_{xy}} \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy.$$

43. The surface in the first octant cut from the cylinder $y = (2/3)z^{3/2}$ by the planes $x = 1$ and $y = 16/3$
44. The portion of the plane $y + z = 4$ that lies above the region cut from the first quadrant of the xz -plane by the parabola $x = 4 - z^2$