# 16.8 The Divergence Theorem and a Unified Theory

The divergence form of Green's Theorem in the plane states that the net outward flux of a vector field across a simple closed curve can be calculated by integrating the divergence of the field over the region enclosed by the curve. The corresponding theorem in three dimensions, called the Divergence Theorem, states that the net outward flux of a vector field across a closed surface in space can be calculated by integrating the divergence of the field over the region enclosed by the surface. In this section, we prove the Divergence Theorem and show how it simplifies the calculation of flux. We also derive Gauss's law for flux in an electric field and the continuity equation of hydrodynamics. Finally, we unify the chapter's vector integral theorems into a single fundamental theorem.

## Divergence in Three Dimensions

The **divergence** of a vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is the scalar function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \tag{1}$$

The symbol "div **F**" is read as "divergence of **F**" or "div **F**." The notation $\nabla \cdot \mathbf{F}$ is read "del dot **F**."

Div **F** has the same physical interpretation in three dimensions that it does in two. If **F** is the velocity field of a fluid flow, the value of div **F** at a point $(x, y, z)$ is the rate at which fluid is being piped in or drained away at $(x, y, z)$. The divergence is the flux per unit volume or flux density at the point.

**EXAMPLE 1** Finding Divergence

Find the divergence of $\mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z\mathbf{k}$.

**Solution** The divergence of **F** is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(-z) = 2z - x - 1.$$

■

## Divergence Theorem

The Divergence Theorem says that under suitable conditions, the outward flux of a vector field across a closed surface (oriented outward) equals the triple integral of the divergence of the field over the region enclosed by the surface.

**THEOREM 7 Divergence Theorem**
The flux of a vector field $\mathbf{F}$ across a closed oriented surface $S$ in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the integral of $\nabla \cdot \mathbf{F}$ over the region $D$ enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV.}_{\text{Divergence integral}} \qquad (2)$$

### EXAMPLE 2 Supporting the Divergence Theorem

Evaluate both sides of Equation (2) for the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$.

**Solution** The outer unit normal to $S$, calculated from the gradient of $f(x, y, z) = x^2 + y^2 + z^2 - a^2$, is

$$\mathbf{n} = \frac{2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

Hence,

$$\mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{x^2 + y^2 + z^2}{a}\, d\sigma = \frac{a^2}{a}\, d\sigma = a\, d\sigma$$

because $x^2 + y^2 + z^2 = a^2$ on the surface. Therefore,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S a\, d\sigma = a \iint_S d\sigma = a(4\pi a^2) = 4\pi a^3.$$

The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

so

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3. \qquad \blacksquare$$

### EXAMPLE 3 Finding Flux

You Try It

Find the flux of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

**Solution** Instead of calculating the flux as a sum of six separate integrals, one for each face of the cube, we can calculate the flux by integrating the divergence

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

over the cube's interior:

$$\text{Flux} = \iint_{\substack{\text{Cube}\\\text{surface}}} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_{\substack{\text{Cube}\\\text{interior}}} \nabla \cdot \mathbf{F}\, dV \qquad \text{The Divergence Theorem}$$

$$= \int_0^1 \int_0^1 \int_0^1 (x + y + z)\, dx\, dy\, dz = \frac{3}{2}. \qquad \text{Routine integration}$$

■

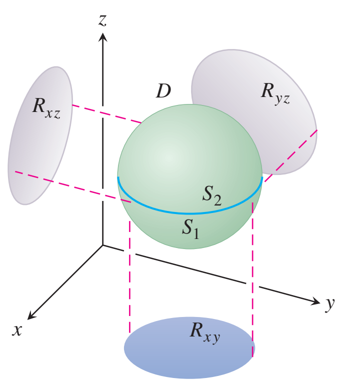


**FIGURE 16.69** We first prove the Divergence Theorem for the kind of three-dimensional region shown here. We then extend the theorem to other regions.

## Proof of the Divergence Theorem for Special Regions

To prove the Divergence Theorem, we assume that the components of **F** have continuous first partial derivatives. We also assume that $D$ is a convex region with no holes or bubbles, such as a solid sphere, cube, or ellipsoid, and that $S$ is a piecewise smooth surface. In addition, we assume that any line perpendicular to the $xy$-plane at an interior point of the region $R_{xy}$ that is the projection of $D$ on the $xy$-plane intersects the surface $S$ in exactly two points, producing surfaces

$$S_1: \quad z = f_1(x, y), \qquad (x, y) \text{ in } R_{xy}$$

$$S_2: \quad z = f_2(x, y), \qquad (x, y) \text{ in } R_{xy},$$

with $f_1 \le f_2$. We make similar assumptions about the projection of $D$ onto the other coordinate planes. See Figure 16.69.

The components of the unit normal vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ are the cosines of the angles $\alpha$, $\beta$, and $\gamma$ that **n** makes with **i**, **j**, and **k** (Figure 16.70). This is true because all the vectors involved are unit vectors. We have

$$n_1 = \mathbf{n} \cdot \mathbf{i} = |\mathbf{n}||\mathbf{i}| \cos \alpha = \cos \alpha$$

$$n_2 = \mathbf{n} \cdot \mathbf{j} = |\mathbf{n}||\mathbf{j}| \cos \beta = \cos \beta$$

$$n_3 = \mathbf{n} \cdot \mathbf{k} = |\mathbf{n}||\mathbf{k}| \cos \gamma = \cos \gamma$$

Thus,

$$\mathbf{n} = (\cos \alpha)\mathbf{i} + (\cos \beta)\mathbf{j} + (\cos \gamma)\mathbf{k}$$

and

$$\mathbf{F} \cdot \mathbf{n} = M \cos \alpha + N \cos \beta + P \cos \gamma.$$

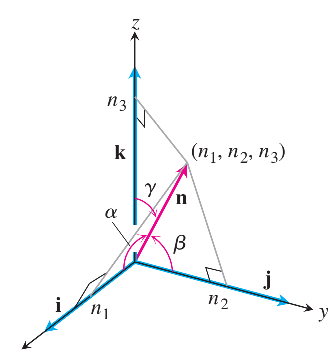


**FIGURE 16.70** The scalar components of the unit normal vector **n** are the cosines of the angles $\alpha$, $\beta$, and $\gamma$ that it makes with **i**, **j**, and **k**.

In component form, the Divergence Theorem states that

$$\iint_S (M \cos \alpha + N \cos \beta + P \cos \gamma)\, d\sigma = \iiint_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) dx\, dy\, dz.$$

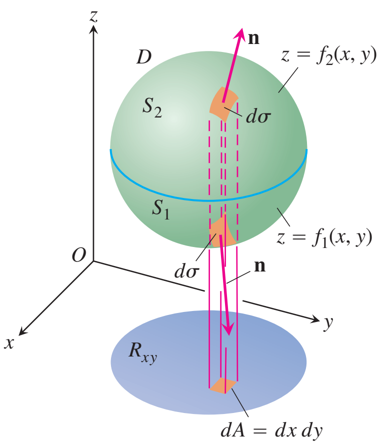


**FIGURE 16.71** The three-dimensional region $D$ enclosed by the surfaces $S_1$ and $S_2$ shown here projects vertically onto a two-dimensional region $R_{xy}$ in the $xy$-plane.

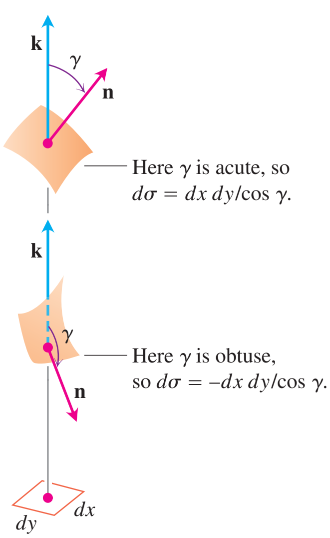


**FIGURE 16.72** An enlarged view of the area patches in Figure 16.71. The relations $d\sigma = \pm dx\,dy/\cos\gamma$ are derived in Section 16.5.

We prove the theorem by proving the three following equalities:

$$\iint_S M\cos\alpha\,d\sigma = \iiint_D \frac{\partial M}{\partial x}\,dx\,dy\,dz \tag{3}$$

$$\iint_S N\cos\beta\,d\sigma = \iiint_D \frac{\partial N}{\partial y}\,dx\,dy\,dz \tag{4}$$

$$\iint_S P\cos\gamma\,d\sigma = \iiint_D \frac{\partial P}{\partial z}\,dx\,dy\,dz \tag{5}$$

**Proof of Equation (5)** We prove Equation (5) by converting the surface integral on the left to a double integral over the projection $R_{xy}$ of $D$ on the $xy$-plane (Figure 16.71). The surface $S$ consists of an upper part $S_2$ whose equation is $z = f_2(x, y)$ and a lower part $S_1$ whose equation is $z = f_1(x, y)$. On $S_2$, the outer normal **n** has a positive **k**-component and

$$\cos\gamma\,d\sigma = dx\,dy \qquad \text{because} \qquad d\sigma = \frac{dA}{|\cos\gamma|} = \frac{dx\,dy}{\cos\gamma}.$$

See Figure 16.72. On $S_1$, the outer normal **n** has a negative **k**-component and

$$\cos\gamma\,d\sigma = -dx\,dy.$$

Therefore,

$$\begin{aligned}\iint_S P\cos\gamma\,d\sigma &= \iint_{S_2} P\cos\gamma\,d\sigma + \iint_{S_1} P\cos\gamma\,d\sigma \\ &= \iint_{R_{xy}} P(x, y, f_2(x, y))\,dx\,dy - \iint_{R_{xy}} P(x, y, f_1(x, y))\,dx\,dy \\ &= \iint_{R_{xy}} [P(x, y, f_2(x, y)) - P(x, y, f_1(x, y))]\,dx\,dy \\ &= \iint_{R_{xy}} \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial P}{\partial z}\,dz\right] dx\,dy = \iiint_D \frac{\partial P}{\partial z}\,dz\,dx\,dy.\end{aligned}$$

This proves Equation (5). ■

The proofs for Equations (3) and (4) follow the same pattern; or just permute $x, y, z$; $M, N, P$; $\alpha, \beta, \gamma$, in order, and get those results from Equation (5).

## Divergence Theorem for Other Regions

The Divergence Theorem can be extended to regions that can be partitioned into a finite number of simple regions of the type just discussed and to regions that can be defined as limits of simpler regions in certain ways. For example, suppose that $D$ is the region between two concentric spheres and that **F** has continuously differentiable components throughout $D$ and on the bounding surfaces. Split $D$ by an equatorial plane and apply the

Divergence Theorem to each half separately. The bottom half, $D_1$, is shown in Figure 16.73. The surface $S_1$ that bounds $D_1$ consists of an outer hemisphere, a plane washer-shaped base, and an inner hemisphere. The Divergence Theorem says that

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 \, d\sigma_1 = \iiint_{D_1} \nabla \cdot \mathbf{F} \, dV_1. \tag{6}$$

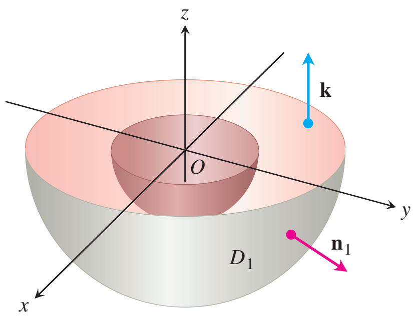

**FIGURE 16.73** The lower half of the solid region between two concentric spheres.

The unit normal $\mathbf{n}_1$ that points outward from $D_1$ points away from the origin along the outer surface, equals $\mathbf{k}$ along the flat base, and points toward the origin along the inner surface. Next apply the Divergence Theorem to $D_2$, and its surface $S_2$ (Figure 16.74):

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, d\sigma_2 = \iiint_{D_2} \nabla \cdot \mathbf{F} \, dV_2. \tag{7}$$

As we follow $\mathbf{n}_2$ over $S_2$, pointing outward from $D_2$, we see that $\mathbf{n}_2$ equals $-\mathbf{k}$ along the washer-shaped base in the $xy$-plane, points away from the origin on the outer sphere, and points toward the origin on the inner sphere. When we add Equations (6) and (7), the integrals over the flat base cancel because of the opposite signs of $\mathbf{n}_1$ and $\mathbf{n}_2$. We thus arrive at the result

$$\iint_{S} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_{D} \nabla \cdot \mathbf{F} \, dV,$$

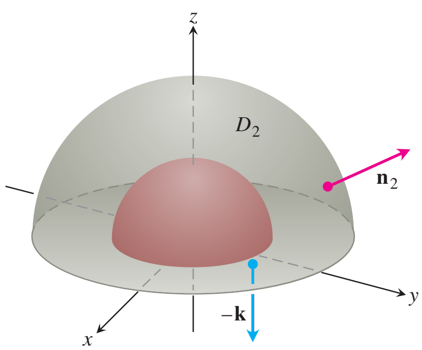

**FIGURE 16.74** The upper half of the solid region between two concentric spheres.

with $D$ the region between the spheres, $S$ the boundary of $D$ consisting of two spheres, and $\mathbf{n}$ the unit normal to $S$ directed outward from $D$.

## EXAMPLE 4 Finding Outward Flux

Find the net outward flux of the field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}, \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region $D$: $0 < a^2 \le x^2 + y^2 + z^2 \le b^2$.

**Solution** The flux can be calculated by integrating $\nabla \cdot \mathbf{F}$ over $D$. We have

$$\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\rho}$$

and

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(x\rho^{-3}) = \rho^{-3} - 3x\rho^{-4}\frac{\partial \rho}{\partial x} = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}.$$

Similarly,

$$\frac{\partial N}{\partial y} = \frac{1}{\rho^3} - \frac{3y^2}{\rho^5} \quad \text{and} \quad \frac{\partial P}{\partial z} = \frac{1}{\rho^3} - \frac{3z^2}{\rho^5}.$$

Hence,

$$\text{div}\,\mathbf{F} = \frac{3}{\rho^3} - \frac{3}{\rho^5}(x^2 + y^2 + z^2) = \frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5} = 0$$

and

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 0.$$

So the integral of $\nabla \cdot \mathbf{F}$ over $D$ is zero and the net outward flux across the boundary of $D$ is zero. There is more to learn from this example, though. The flux leaving $D$ across the inner sphere $S_a$ is the negative of the flux leaving $D$ across the outer sphere $S_b$ (because the sum of these fluxes is zero). Hence, the flux of $\mathbf{F}$ across $S_a$ in the direction away from the origin equals the flux of $\mathbf{F}$ across $S_b$ in the direction away from the origin. Thus, the flux of $\mathbf{F}$ across a sphere centered at the origin is independent of the radius of the sphere. What is this flux?

To find it, we evaluate the flux integral directly. The outward unit normal on the sphere of radius $a$ is

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

Hence, on the sphere,

$$\mathbf{F} \cdot \mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a^3} \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} = \frac{x^2 + y^2 + z^2}{a^4} = \frac{a^2}{a^4} = \frac{1}{a^2}$$

and

$$\iint_{S_a} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{1}{a^2} \iint_{S_a} d\sigma = \frac{1}{a^2}(4\pi a^2) = 4\pi.$$

The outward flux of $\mathbf{F}$ across any sphere centered at the origin is $4\pi$. ■

## Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

There is still more to be learned from Example 4. In electromagnetic theory, the electric field created by a point charge $q$ located at the origin is

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r}|^2} \left( \frac{\mathbf{r}}{|\mathbf{r}|} \right) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{q}{4\pi\epsilon_0} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3},$$

where $\epsilon_0$ is a physical constant, $\mathbf{r}$ is the position vector of the point $(x, y, z)$, and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. In the notation of Example 4,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \mathbf{F}.$$

The calculations in Example 4 show that the outward flux of $\mathbf{E}$ across any sphere centered at the origin is $q/\epsilon_0$, but this result is not confined to spheres. The outward flux of $\mathbf{E}$ across any closed surface $S$ that encloses the origin (and to which the Divergence Theorem applies) is also $q/\epsilon_0$. To see why, we have only to imagine a large sphere $S_a$ centered at the origin and enclosing the surface $S$. Since

$$\nabla \cdot \mathbf{E} = \nabla \cdot \frac{q}{4\pi\epsilon_0} \mathbf{F} = \frac{q}{4\pi\epsilon_0} \nabla \cdot \mathbf{F} = 0$$

when $\rho > 0$, the integral of $\nabla \cdot \mathbf{E}$ over the region $D$ between $S$ and $S_a$ is zero. Hence, by the Divergence Theorem,

$$\iint_{\substack{\text{Boundary}\\ \text{of } D}} \mathbf{E} \cdot \mathbf{n}\, d\sigma = 0,$$

and the flux of $\mathbf{E}$ across $S$ in the direction away from the origin must be the same as the flux of $\mathbf{E}$ across $S_a$ in the direction away from the origin, which is $q/\epsilon_0$. This statement, called *Gauss's Law*, also applies to charge distributions that are more general than the one assumed here, as you will see in nearly any physics text.

$$\text{Gauss's law:} \quad \iint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \frac{q}{\epsilon_0}$$

## Continuity Equation of Hydrodynamics

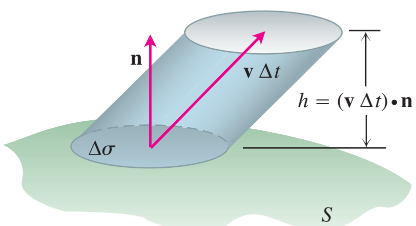


**FIGURE 16.75** The fluid that flows upward through the patch $\Delta\sigma$ in a short time $\Delta t$ fills a "cylinder" whose volume is approximately base $\times$ height $= \mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t$.

Let $D$ be a region in space bounded by a closed oriented surface $S$. If $\mathbf{v}(x, y, z)$ is the velocity field of a fluid flowing smoothly through $D$, $\delta = \delta(t, x, y, z)$ is the fluid's density at $(x, y, z)$ at time $t$, and $\mathbf{F} = \delta\mathbf{v}$, then the **continuity equation** of hydrodynamics states that

$$\nabla \cdot \mathbf{F} + \frac{\partial \delta}{\partial t} = 0.$$

If the functions involved have continuous first partial derivatives, the equation evolves naturally from the Divergence Theorem, as we now see.

First, the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

is the rate at which mass leaves $D$ across $S$ (leaves because $\mathbf{n}$ is the outer normal). To see why, consider a patch of area $\Delta\sigma$ on the surface (Figure 16.75). In a short time interval $\Delta t$, the volume $\Delta V$ of fluid that flows across the patch is approximately equal to the volume of a cylinder with base area $\Delta\sigma$ and height $(\mathbf{v}\Delta t) \cdot \mathbf{n}$, where $\mathbf{v}$ is a velocity vector rooted at a point of the patch:

$$\Delta V \approx \mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t.$$

The mass of this volume of fluid is about

$$\Delta m \approx \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t,$$

so the rate at which mass is flowing out of $D$ across the patch is about

$$\frac{\Delta m}{\Delta t} \approx \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma.$$

This leads to the approximation

$$\frac{\sum \Delta m}{\Delta t} \approx \sum \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma$$

as an estimate of the average rate at which mass flows across $S$. Finally, letting $\Delta\sigma \to 0$ and $\Delta t \to 0$ gives the instantaneous rate at which mass leaves $D$ across $S$ as

$$\frac{dm}{dt} = \iint_S \delta \mathbf{v} \cdot \mathbf{n}\, d\sigma,$$

which for our particular flow is

$$\frac{dm}{dt} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Now let $B$ be a solid sphere centered at a point $Q$ in the flow. The average value of $\nabla \cdot \mathbf{F}$ over $B$ is

$$\frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F}\, dV.$$

It is a consequence of the continuity of the divergence that $\nabla \cdot \mathbf{F}$ actually takes on this value at some point $P$ in $B$. Thus,

$$(\nabla \cdot \mathbf{F})_P = \frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F}\, dV = \frac{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}{\text{volume of } B}$$

$$= \frac{\text{rate at which mass leaves } B \text{ across its surface } S}{\text{volume of } B} \tag{8}$$

The fraction on the right describes decrease in mass per unit volume.

Now let the radius of $B$ approach zero while the center $Q$ stays fixed. The left side of Equation (8) converges to $(\nabla \cdot \mathbf{F})_Q$, the right side to $(-\partial\delta/\partial t)_Q$. The equality of these two limits is the continuity equation

$$\nabla \cdot \mathbf{F} = -\frac{\partial \delta}{\partial t}.$$

The continuity equation "explains" $\nabla \cdot \mathbf{F}$: The divergence of $\mathbf{F}$ at a point is the rate at which the density of the fluid is decreasing there.

The Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

now says that the net decrease in density of the fluid in region $D$ is accounted for by the mass transported across the surface $S$. So, the theorem is a statement about conservation of mass (Exercise 31).

## Unifying the Integral Theorems

If we think of a two-dimensional field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ as a three-dimensional field whose $\mathbf{k}$-component is zero, then $\nabla \cdot \mathbf{F} = (\partial M/\partial x) + (\partial N/\partial y)$ and the normal form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R \nabla \cdot \mathbf{F}\, dA.$$

Similarly, $\nabla \times \mathbf{F} \cdot \mathbf{k} = (\partial N/\partial x) - (\partial M/\partial y)$, so the tangential form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA.$$

With the equations of Green's Theorem now in del notation, we can see their relationships to the equations in Stokes' Theorem and the Divergence Theorem.

### Green's Theorem and Its Generalization to Three Dimensions

**Normal form of Green's Theorem:** $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$

**Divergence Theorem:** $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$

**Tangential form of Green's Theorem:** $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$

**Stokes' Theorem:** $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

Notice how Stokes' Theorem generalizes the tangential (curl) form of Green's Theorem from a flat surface in the plane to a surface in three-dimensional space. In each case, the integral of the normal component of curl **F** over the interior of the surface equals the circulation of **F** around the boundary.

Likewise, the Divergence Theorem generalizes the normal (flux) form of Green's Theorem from a two-dimensional region in the plane to a three-dimensional region in space. In each case, the integral of $\nabla \cdot \mathbf{F}$ over the interior of the region equals the total flux of the field across the boundary.

There is still more to be learned here. All these results can be thought of as forms of a *single fundamental theorem*. Think back to the Fundamental Theorem of Calculus in Section 5.3. It says that if $f(x)$ is differentiable on $(a, b)$ and continuous on $[a, b]$, then


**FIGURE 16.76** The outward unit normals at the boundary of $[a, b]$ in one-dimensional space.

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a).$$

If we let $\mathbf{F} = f(x)\mathbf{i}$ throughout $[a, b]$, then $(df/dx) = \nabla \cdot \mathbf{F}$. If we define the unit vector field **n** normal to the boundary of $[a, b]$ to be **i** at $b$ and $-\mathbf{i}$ at $a$ (Figure 16.76), then

$$\begin{aligned} f(b) - f(a) &= f(b)\mathbf{i} \cdot (\mathbf{i}) + f(a)\mathbf{i} \cdot (-\mathbf{i}) \\ &= \mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} \\ &= \text{total outward flux of } \mathbf{F} \text{ across the boundary of } [a, b]. \end{aligned}$$

The Fundamental Theorem now says that

$$\mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} = \int_{[a,b]} \nabla \cdot \mathbf{F}\, dx.$$

The Fundamental Theorem of Calculus, the normal form of Green's Theorem, and the Divergence Theorem all say that the integral of the differential operator $\nabla \cdot$ operating on a field **F** over a region equals the sum of the normal field components over the boundary of the region. (Here we are interpreting the line integral in Green's Theorem and the surface integral in the Divergence Theorem as "sums" over the boundary.)

Stokes' Theorem and the tangential form of Green's Theorem say that, when things are properly oriented, the integral of the normal component of the curl operating on a field equals the sum of the tangential field components on the boundary of the surface.

The beauty of these interpretations is the observance of a single unifying principle, which we might state as follows.

> The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.