A.2 Proofs of Limit Theorems

This appendix proves Theorem 1, Parts 2–5, and Theorem 4 from Section 2.2.

THEOREM 1 Limit Laws

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}\left(f(x) + g(x)\right) = L + M$
2. *Difference Rule:* $\lim_{x\to c}\left(f(x) - g(x)\right) = L - M$
3. *Product Rule:* $\lim_{x\to c}\left(f(x)\cdot g(x)\right) = L\cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to c}\left(kf(x)\right) = kL \quad$ (any number k)
5. *Quotient Rule:* $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad$ if $M \neq 0$
6. *Power Rule:* If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x\to c}\left(f(x)\right)^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

We proved the Sum Rule in Section 2.3 and the Power Rule is proved in more advanced texts. We obtain the Difference Rule by replacing $g(x)$ by $-g(x)$ and M by $-M$ in the Sum Rule. The Constant Multiple Rule is the special case $g(x) = k$ of the Product Rule. This leaves only the Product and Quotient Rules.

Proof of the Limit Product Rule We show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all x in the intersection D of the domains of f and g,

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x)g(x) - LM| < \epsilon.$$

Suppose then that ϵ is a positive number, and write $f(x)$ and $g(x)$ as

$$f(x) = L + (f(x) - L), \qquad g(x) = M + (g(x) - M).$$

Multiply these expressions together and subtract LM:

$$\begin{aligned} f(x)\cdot g(x) - LM &= (L + (f(x) - L))(M + (g(x) - M)) - LM \\ &= LM + L(g(x) - M) + M(f(x) - L) \\ &\quad + (f(x) - L)(g(x) - M) - LM \\ &= L(g(x) - M) + M(f(x) - L) + (f(x) - L)(g(x) - M). \quad (1) \end{aligned}$$

Since f and g have limits L and M as $x \to c$, there exist positive numbers δ_1, δ_2, δ_3, and δ_4 such that for all x in D

$$\begin{aligned} 0 < |x - c| < \delta_1 &\quad\Rightarrow\quad |f(x) - L| < \sqrt{\epsilon/3} \\ 0 < |x - c| < \delta_2 &\quad\Rightarrow\quad |g(x) - M| < \sqrt{\epsilon/3} \\ 0 < |x - c| < \delta_3 &\quad\Rightarrow\quad |f(x) - L| < \epsilon/(3(1 + |M|)) \\ 0 < |x - c| < \delta_4 &\quad\Rightarrow\quad |g(x) - M| < \epsilon/(3(1 + |L|)) \end{aligned} \quad (2)$$

If we take δ to be the smallest numbers δ_1 through δ_4, the inequalities on the right-hand side of the Implications (2) will hold simultaneously for $0 < |x - c| < \delta$. Therefore, for all x in D, $0 < |x - c| < \delta$ implies

$|f(x)\cdot g(x) - LM|$ — Triangle inequality applied to Equation (1)

$$\begin{aligned} &\le |L||g(x) - M| + |M||f(x) - L| + |f(x) - L||g(x) - M| \\ &\le (1 + |L|)|g(x) - M| + (1 + |M|)|f(x) - L| + |f(x) - L||g(x) - M| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \sqrt{\frac{\epsilon}{3}}\sqrt{\frac{\epsilon}{3}} = \epsilon. \end{aligned}$$

Values from (2)

This completes the proof of the Limit Product Rule. ■

Proof of the Limit Quotient Rule We show that $\lim_{x\to c}(1/g(x)) = 1/M$. We can then conclude that

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\left(f(x)\cdot\frac{1}{g(x)}\right) = \lim_{x\to c} f(x)\cdot\lim_{x\to c}\frac{1}{g(x)} = L\cdot\frac{1}{M} = \frac{L}{M}$$

by the Limit Product Rule.

Let $\epsilon > 0$ be given. To show that $\lim_{x\to c}(1/g(x)) = 1/M$, we need to show that there exists a $\delta > 0$ such that for all x.

$$0 < |x - c| < \delta \quad\Rightarrow\quad \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon.$$

Since $|M| > 0$, there exists a positive number δ_1 such that for all x

$$0 < |x - c| < \delta_1 \quad\Rightarrow\quad |g(x) - M| < \frac{M}{2}. \quad (3)$$

For any numbers A and B it can be shown that $|A| - |B| \le |A - B|$ and $|B| - |A| \le |A - B|$, from which it follows that $||A| - |B|| \le |A - B|$. With $A = g(x)$ and $B = M$, this becomes

$$||g(x)| - |M|| \le |g(x) - M|,$$

which can be combined with the inequality on the right in Implication (3) to get, in turn,

$$\big|\,|g(x)| - |M|\,\big| < \frac{|M|}{2}$$

$$-\frac{|M|}{2} < |g(x)| - |M| < \frac{|M|}{2}$$

$$\frac{|M|}{2} < |g(x)| < \frac{3|M|}{2}$$

$$|M| < 2|g(x)| < 3|M|$$

$$\frac{1}{|g(x)|} < \frac{2}{|M|} < \frac{3}{|g(x)|} \tag{4}$$

Therefore, $0 < |x - c| < \delta_1$ implies that

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{Mg(x)}\right| \le \frac{1}{|M|}\cdot\frac{1}{|g(x)|}\cdot|M - g(x)|$$

$$< \frac{1}{|M|}\cdot\frac{2}{|M|}\cdot|M - g(x)|. \quad \text{Inequality (4)} \tag{5}$$

Since $(1/2)|M|^2\epsilon > 0$, there exists a number $\delta_2 > 0$ such that for all x

$$0 < |x - c| < \delta_2 \quad \Rightarrow \quad |M - g(x)| < \frac{\epsilon}{2}|M|^2. \tag{6}$$

If we take δ to be the smaller of δ_1 and δ_2, the conclusions in (5) and (6) both hold for all x such that $0 < |x - c| < \delta$. Combining these conclusions gives

$$0 < |x - c| < \delta \quad \Rightarrow \quad \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon.$$

This concludes the proof of the Limit Quotient Rule. ■

THEOREM 4 The Sandwich Theorem
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval I containing c, except possibly at $x = c$ itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then $\lim_{x\to c} f(x) = L$.

Proof for Right-Hand Limits Suppose $\lim_{x\to c^+} g(x) = \lim_{x\to c^+} h(x) = L$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all x the interval $c < x < c + \delta$ is contained in I and the inequality implies

$$L - \epsilon < g(x) < L + \epsilon \quad \text{and} \quad L - \epsilon < h(x) < L + \epsilon.$$

These inequalities combine with the inequality $g(x) \le f(x) \le h(x)$ to give

$$L - \epsilon < g(x) \le f(x) \le h(x) < L + \epsilon,$$

$$L - \epsilon < f(x) < L + \epsilon,$$

$$-\epsilon < f(x) - L < \epsilon.$$

Therefore, for all x, the inequality $c < x < c + \delta$ implies $|f(x) - L| < \epsilon$. ■

Proof for Left-Hand Limits Suppose $\lim_{x \to c^-} g(x) = \lim_{x \to c^-} h(x) = L$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all x the interval $c - \delta < x < c$ is contained in I and the inequality implies

$$L - \epsilon < g(x) < L + \epsilon \quad \text{and} \quad L - \epsilon < h(x) < L + \epsilon.$$

We conclude as before that for all x, $c - \delta < x < c$ implies $|f(x) - L| < \epsilon$. ■

Proof for Two-Sided Limits If $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then $g(x)$ and $h(x)$ both approach L as $x \to c^+$ and as $x \to c^-$; so $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$. Hence $\lim_{x \to c} f(x)$ exists and equals L. ■

EXERCISES A.2

1. Suppose that functions $f_1(x)$, $f_2(x)$, and $f_3(x)$ have limits L_1, L_2, and L_3, respectively, as $x \to c$. Show that their sum has limit $L_1 + L_2 + L_3$. Use mathematical induction (Appendix 1) to generalize this result to the sum of any finite number of functions.

2. Use mathematical induction and the Limit Product Rule in Theorem 1 to show that if functions $f_1(x), f_2(x), \dots, f_n(x)$ have limits $L_1, L_2, \dots, L_n$ as $x \to c$, then

$$\lim_{x \to c} f_1(x) f_2(x) \cdot \cdots \cdot f_n(x) = L_1 \cdot L_2 \cdot \cdots \cdot L_n.$$

3. Use the fact that $\lim_{x \to c} x = c$ and the result of Exercise 2 to show that $\lim_{x \to c} x^n = c^n$ for any integer $n > 1$.

4. **Limits of polynomials** Use the fact that $\lim_{x \to c}(k) = k$ for any number k together with the results of Exercises 1 and 3 to show that $\lim_{x \to c} f(x) = f(c)$ for any polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

5. **Limits of rational functions** Use Theorem 1 and the result of Exercise 4 to show that if $f(x)$ and $g(x)$ are polynomial functions and $g(c) \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

6. **Composites of continuous functions** Figure A.1 gives the diagram for a proof that the composite of two continuous functions is continuous. Reconstruct the proof from the diagram. The statement to be proved is this: If f is continuous at $x = c$ and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

   Assume that c is an interior point of the domain of f and that $f(c)$ is an interior point of the domain of g. This will make the limits involved two-sided. (The arguments for the cases that involve one-sided limits are similar.)

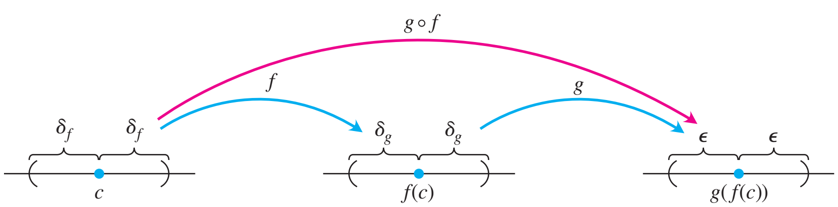

FIGURE A.1 The diagram for a proof that the composite of two continuous functions is continuous.