A.8 The Area of a Parallelogram's Projection on a Plane

This appendix proves the result needed in Section 16.5 that $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{p}|$ is the area of the projection of the parallelogram with sides determined by $\mathbf{u}$ and $\mathbf{v}$ onto any plane whose normal is $\mathbf{p}$. (See Figure A.15.)

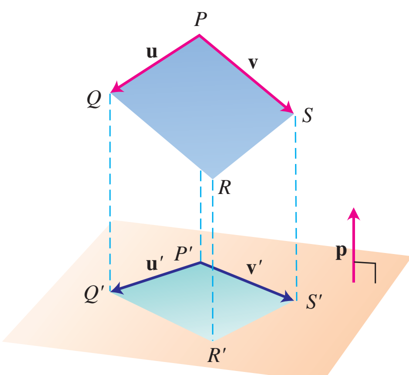

FIGURE A.15 The parallelogram determined by two vectors **u** and **v** in space and the orthogonal projection of the parallelogram onto a plane. The projection lines, orthogonal to the plane, lie parallel to the unit normal vector **p**.

THEOREM

The area of the orthogonal projection of the parallelogram determined by two vectors **u** and **v** in space onto a plane with unit normal vector **p** is

$$\text{Area} = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{p}|.$$

Proof In the notation of Figure A.15, which shows a typical parallelogram determined by vectors **u** and **v** and its orthogonal projection onto a plane with unit normal vector **p**,

$$\begin{aligned} \mathbf{u} &= \overrightarrow{PP'} + \mathbf{u}' + \overrightarrow{Q'Q} \\ &= \mathbf{u}' + \overrightarrow{PP'} - \overrightarrow{QQ'} \qquad (\overrightarrow{Q'Q} = -\overrightarrow{QQ'}) \\ &= \mathbf{u}' + s\mathbf{p}. \end{aligned}$$

(For some scalar s because $(\overrightarrow{PP'} - \overrightarrow{QQ'})$ is parallel to **p**)

Similarly,

$$\mathbf{v} = \mathbf{v}' + t\mathbf{p}$$

for some scalar t. Hence,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (\mathbf{u}' + s\mathbf{p}) \times (\mathbf{v}' + t\mathbf{p}) \\ &= (\mathbf{u}' \times \mathbf{v}') + s(\mathbf{p} \times \mathbf{v}') + t(\mathbf{u}' \times \mathbf{p}) + st\underbrace{(\mathbf{p} \times \mathbf{p})}_{0}. \end{aligned} \qquad (1)$$

The vectors $\mathbf{p} \times \mathbf{v}'$ and $\mathbf{u}' \times \mathbf{p}$ are both orthogonal to **p**. Hence, when we dot both sides of Equation (1) with **p**, the only nonzero term on the right is $(\mathbf{u}' \times \mathbf{v}') \cdot \mathbf{p}$. That is,

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{p} = (\mathbf{u}' \times \mathbf{v}') \cdot \mathbf{p}.$$

In particular,

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{p}| = |(\mathbf{u}' \times \mathbf{v}') \cdot \mathbf{p}|. \qquad (2)$$

The absolute value on the right is the volume of the box determined by $\mathbf{u}'$, $\mathbf{v}'$, and **p**. The height of this particular box is $|\mathbf{p}| = 1$, so the box's volume is numerically the same as its base area, the area of parallelogram $P'Q'R'S'$. Combining this observation with Equation (2) gives

$$\text{Area of } P'Q'R'S' = |(\mathbf{u}' \times \mathbf{v}') \cdot \mathbf{p}| = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{p}|,$$

which says that the area of the orthogonal projection of the parallelogram determined by **u** and **v** onto a plane with unit normal vector **p** is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{p}|$. This is what we set out to prove. ■

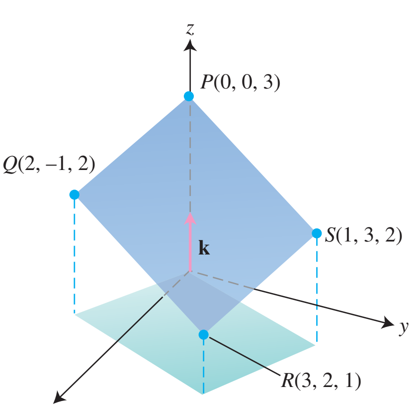

FIGURE A.16 Example 1 calculates the area of the orthogonal projection of parallelogram $PQRS$ on the xy-plane.

EXAMPLE 1 Finding the Area of a Projection

Find the area of the orthogonal projection onto the xy-plane of the parallelogram determined by the points $P(0, 0, 3)$, $Q(2, -1, 2)$, $R(3, 2, 1)$, and $S(1, 3, 2)$ (Figure A.16).

Solution With

$$\mathbf{u} = \overrightarrow{PQ} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}, \qquad \mathbf{v} = \overrightarrow{PS} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}, \qquad \text{and} \qquad \mathbf{p} = \mathbf{k},$$

we have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{p} = \begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7,$$

so the area is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{p}| = |7| = 7.$ ■