EXERCISES 2.2

Limit Calculations

Find the limits in Exercises 1–18.

1. $\lim_{x\to-7}(2x+5)$
2. $\lim_{x\to12}(10-3x)$
3. $\lim_{x\to2}(-x^2+5x-2)$
4. $\lim_{x\to-2}(x^3-2x^2+4x+8)$
5. $\lim_{t\to6}8(t-5)(t-7)$
6. $\lim_{s\to2/3}3s(2s-1)$
7. $\lim_{x\to2}\frac{x+3}{x+6}$
8. $\lim_{x\to5}\frac{4}{x-7}$
9. $\lim_{y\to-5}\frac{y^2}{5-y}$
10. $\lim_{y\to2}\frac{y+2}{y^2+5y+6}$
11. $\lim_{x\to-1}3(2x-1)^2$
12. $\lim_{x\to-4}(x+3)^{1984}$
13. $\lim_{y\to-3}(5-y)^{4/3}$
14. $\lim_{z\to0}(2z-8)^{1/3}$
15. $\lim_{h\to0}\frac{3}{\sqrt{3h+1}+1}$
16. $\lim_{h\to0}\frac{5}{\sqrt{5h+4}+2}$
17. $\lim_{h\to0}\frac{\sqrt{3h+1}-1}{h}$
18. $\lim_{h\to0}\frac{\sqrt{5h+4}-2}{h}$

Find the limits in Exercises 19–36.

19. $\lim_{x\to5}\frac{x-5}{x^2-25}$
20. $\lim_{x\to-3}\frac{x+3}{x^2+4x+3}$
21. $\lim_{x\to-5}\frac{x^2+3x-10}{x+5}$
22. $\lim_{x\to2}\frac{x^2-7x+10}{x-2}$
23. $\lim_{t\to1}\frac{t^2+t-2}{t^2-1}$
24. $\lim_{t\to-1}\frac{t^2+3t+2}{t^2-t-2}$
25. $\lim_{x\to-2}\frac{-2x-4}{x^3+2x^2}$
26. $\lim_{y\to0}\frac{5y^3+8y^2}{3y^4-16y^2}$
27. $\lim_{u\to1}\frac{u^4-1}{u^3-1}$
28. $\lim_{v\to2}\frac{v^3-8}{v^4-16}$
29. $\lim_{x\to9}\frac{\sqrt{x}-3}{x-9}$
30. $\lim_{x\to4}\frac{4x-x^2}{2-\sqrt{x}}$
31. $\lim_{x\to1}\frac{x-1}{\sqrt{x+3}-2}$
32. $\lim_{x\to-1}\frac{\sqrt{x^2+8}-3}{x+1}$
33. $\lim_{x\to2}\frac{\sqrt{x^2+12}-4}{x-2}$
34. $\lim_{x\to-2}\frac{x+2}{\sqrt{x^2+5}-3}$

35. $\lim\limits_{x\to-3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

36. $\lim\limits_{x\to4} \dfrac{4-x}{5-\sqrt{x^2+9}}$

Using Limit Rules

37. Suppose $\lim_{x\to0} f(x) = 1$ and $\lim_{x\to0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to0} \frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim\limits_{x\to0}(2f(x)-g(x))}{\lim\limits_{x\to0}(f(x)+7)^{2/3}} \quad \text{(a)}$$

$$= \frac{\lim\limits_{x\to0} 2f(x) - \lim\limits_{x\to0} g(x)}{\left(\lim\limits_{x\to0}\left(f(x)+7\right)\right)^{2/3}} \quad \text{(b)}$$

$$= \frac{2\lim\limits_{x\to0} f(x) - \lim\limits_{x\to0} g(x)}{\left(\lim\limits_{x\to0} f(x) + \lim\limits_{x\to0} 7\right)^{2/3}} \quad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

38. Let $\lim_{x\to1} h(x) = 5$, $\lim_{x\to1} p(x) = 1$, and $\lim_{x\to1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to1} \frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim\limits_{x\to1}\sqrt{5h(x)}}{\lim\limits_{x\to1}(p(x)(4-r(x)))} \quad \text{(a)}$$

$$= \frac{\sqrt{\lim\limits_{x\to1} 5h(x)}}{\left(\lim\limits_{x\to1} p(x)\right)\left(\lim\limits_{x\to1}\left(4-r(x)\right)\right)} \quad \text{(b)}$$

$$= \frac{\sqrt{5\lim\limits_{x\to1} h(x)}}{\left(\lim\limits_{x\to1} p(x)\right)\left(\lim\limits_{x\to1} 4 - \lim\limits_{x\to1} r(x)\right)} \quad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

39. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim\limits_{x\to c} f(x)g(x)$

b. $\lim\limits_{x\to c} 2f(x)g(x)$

c. $\lim\limits_{x\to c} (f(x) + 3g(x))$

d. $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

40. Suppose $\lim_{x\to4} f(x) = 0$ and $\lim_{x\to4} g(x) = -3$. Find

a. $\lim\limits_{x\to4} (g(x)+3)$

b. $\lim\limits_{x\to4} xf(x)$

c. $\lim\limits_{x\to4} (g(x))^2$

d. $\lim\limits_{x\to4} \dfrac{g(x)}{f(x)-1}$

41. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

a. $\lim\limits_{x\to b} (f(x)+g(x))$

b. $\lim\limits_{x\to b} f(x)\cdot g(x)$

c. $\lim\limits_{x\to b} 4g(x)$

d. $\lim\limits_{x\to b} f(x)/g(x)$

42. Suppose that $\lim_{x\to-2} p(x) = 4$, $\lim_{x\to-2} r(x) = 0$, and $\lim_{x\to-2} s(x) = -3$. Find

a. $\lim\limits_{x\to-2} (p(x)+r(x)+s(x))$

b. $\lim\limits_{x\to-2} p(x)\cdot r(x)\cdot s(x)$

c. $\lim\limits_{x\to-2} (-4p(x)+5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to0} \frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 43–48, evaluate this limit for the given value of x and function f.

43. $f(x) = x^2, \quad x = 1$

44. $f(x) = x^2, \quad x = -2$

45. $f(x) = 3x - 4, \quad x = 2$

46. $f(x) = 1/x, \quad x = -2$

47. $f(x) = \sqrt{x}, \quad x = 7$

48. $f(x) = \sqrt{3x+1}, \quad x = 0$

Using the Sandwich Theorem

49. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to0} f(x)$.

50. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to0} g(x)$.

51. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x\sin x}{2-2\cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to0} \frac{x\sin x}{2-2\cos x}?$$

Give reasons for your answer.

T **b.** Graph

$$y = 1-(x^2/6),\ y = (x\sin x)/(2-2\cos x),\ \text{and } y = 1$$

together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

52. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 11.9.) What, if anything, does this tell you about

$$\lim_{x\to0} \frac{1-\cos x}{x^2}?$$

Give reasons for your answer.

b. Graph the equations $y = (1/2) - (x^2/24)$, $y = (1-\cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

Theory and Examples

53. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x\to c} f(x)$? What can you say about the value of the limit at these points?

54. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x\to 2} f(x) = 0$? Give reasons for your answers.

55. If $\lim_{x\to 4} \dfrac{f(x) - 5}{x - 2} = 1$, find $\lim_{x\to 4} f(x)$.

56. If $\lim_{x\to -2} \dfrac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to -2} f(x)$ **b.** $\lim_{x\to -2} \dfrac{f(x)}{x}$

57. a. If $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 3$, find $\lim_{x\to 2} f(x)$.

b. If $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 4$, find $\lim_{x\to 2} f(x)$.

58. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to 0} f(x)$ **b.** $\lim_{x\to 0} \dfrac{f(x)}{x}$

T **59. a.** Graph $g(x) = x \sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T **60. a.** Graph $h(x) = x^2 \cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.