2.4 One-Sided Limits and Limits at Infinity

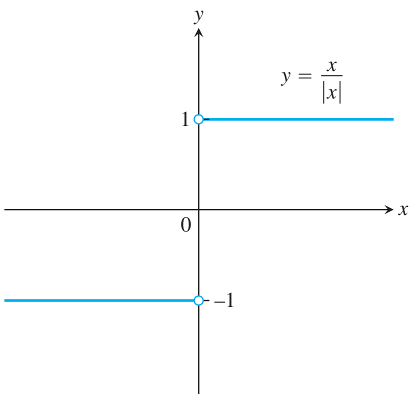


FIGURE 2.21 Different right-hand and left-hand limits at the origin.

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number x_0 from the left-hand side (where $x < x_0$) or the right-hand side $(x > x_0)$ only. We also analyze the graphs of certain rational functions as well as other functions with limit behavior as $x \to \pm\infty$.

One-Sided Limits

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

The function $f(x) = x/|x|$ (Figure 2.21) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

$$\lim_{x \to c^+} f(x) = L.$$

The symbol "$x \to c^+$" means that we consider only values of x greater than c.

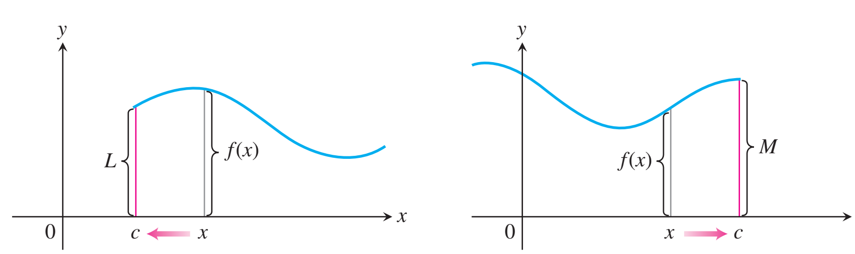
Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider only x values less than c.

These informal definitions are illustrated in Figure 2.22. For the function $f(x) = x/|x|$ in Figure 2.21 we have

$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$



(a) $\lim_{x \to c^+} f(x) = L$ (b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.22 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

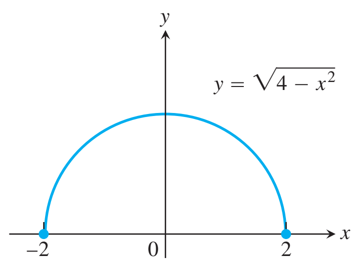


FIGURE 2.23 $\lim\limits_{x \to 2^-} \sqrt{4 - x^2} = 0$ and $\lim\limits_{x \to -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

EXAMPLE 1 One-Sided Limits for a Semicircle

The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.23. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2. ■

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem and Theorem 5. One-sided limits are related to limits in the following way.

THEOREM 6
A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

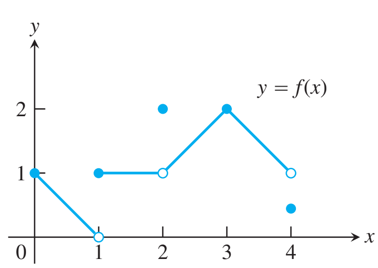


FIGURE 2.24 Graph of the function in Example 2.

EXAMPLE 2 Limits of the Function Graphed in Figure 2.24

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$,
$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,
$\lim_{x \to 1^+} f(x) = 1$,
$\lim_{x \to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,
$\lim_{x \to 2^+} f(x) = 1$,
$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$,
$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

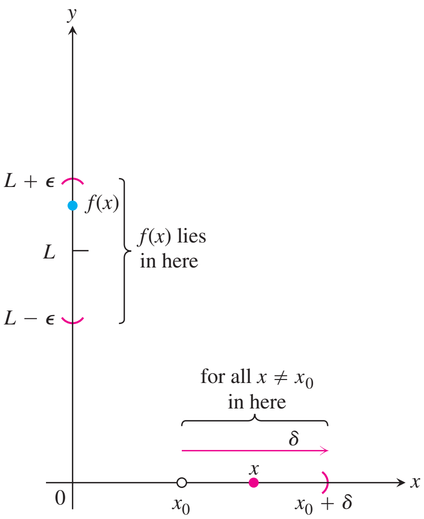


FIGURE 2.25 Intervals associated with the definition of right-hand limit.

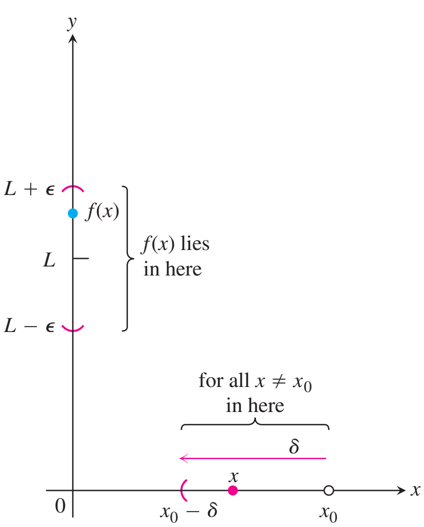


FIGURE 2.26 Intervals associated with the definition of left-hand limit.

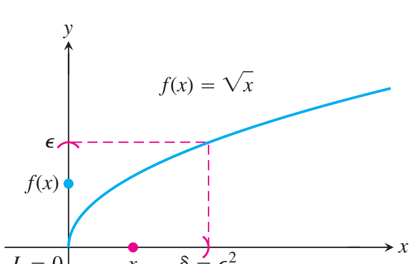


FIGURE 2.27 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

DEFINITIONS Right-Hand, Left-Hand Limits

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

EXAMPLE 3 Applying the Definition to Find Delta

Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\epsilon > 0$ be given. Here $x_0 = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \qquad \Rightarrow \qquad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \qquad \Rightarrow \qquad \sqrt{x} < \epsilon.$$

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \qquad \text{if} \qquad 0 < x < \delta.$$

If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \qquad \Rightarrow \qquad \sqrt{x} < \epsilon,$$

or

$$0 < x < \epsilon^2 \qquad \Rightarrow \qquad |\sqrt{x} - 0| < \epsilon.$$

According to the definition, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$ (Figure 2.27). ■

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 A Function Oscillating Too Much

Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.28).

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's

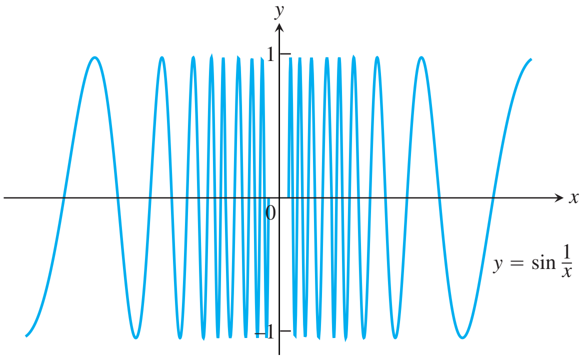


FIGURE 2.28 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4).

values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin \theta)/\theta$

A central fact about $(\sin \theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.29 and confirm it algebraically using the Sandwich Theorem.

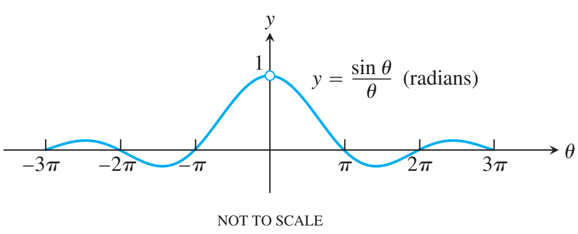


FIGURE 2.29 The graph of $f(\theta) = (\sin \theta)/\theta$.

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

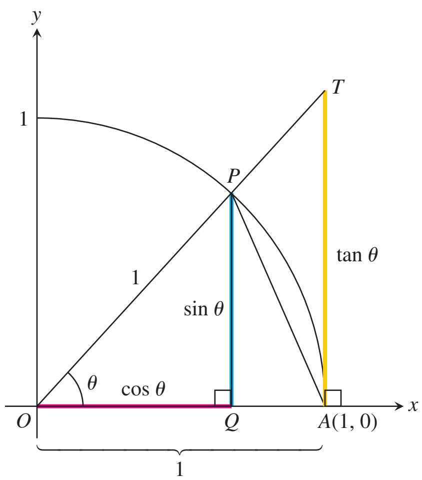


FIGURE 2.30 The figure for the proof of Theorem 7. $TA/OA = \tan \theta$, but $OA = 1$, so $TA = \tan \theta$.

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.30). Notice that

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

Equation (2) is where radian measure comes in: The area of sector OAP is $\theta/2$ only if θ is measured in radians.

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \tag{2}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$ (Example 6b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

Recall that $\sin\theta$ and θ are both *odd functions* (Section 1.4). Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.29). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem 6. ■

EXAMPLE 5 Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Show that **(a)** $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ and **(b)** $\lim_{x\to 0}\frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h} \\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta && \text{Let } \theta = h/2. \\ &= -(1)(0) = 0.\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}\\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

Now, Eq. (1) applies with $\theta = 2x$.

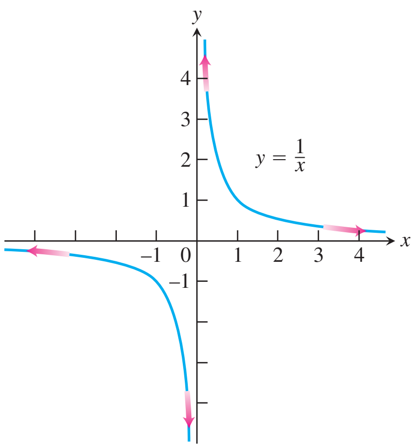


FIGURE 2.31 The graph of $y = 1/x$.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.31). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \pm\infty$ or that 0 is a *limit of* $f(x) = 1/x$ *at infinity and negative infinity*. Here is a precise definition.

DEFINITIONS Limit as x approaches ∞ or $-\infty$

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write
$$\lim_{x\to\infty} f(x) = L$$
if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x
$$x > M \quad\Rightarrow\quad |f(x) - L| < \epsilon.$$
2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write
$$\lim_{x\to-\infty} f(x) = L$$
if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x
$$x < N \quad\Rightarrow\quad |f(x) - L| < \epsilon.$$

Intuitively, $\lim_{x\to\infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly, $\lim_{x\to-\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

The strategy for calculating limits of functions as $x \to \pm\infty$ is similar to the one for finite limits in Section 2.2. There we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying a theorem about limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

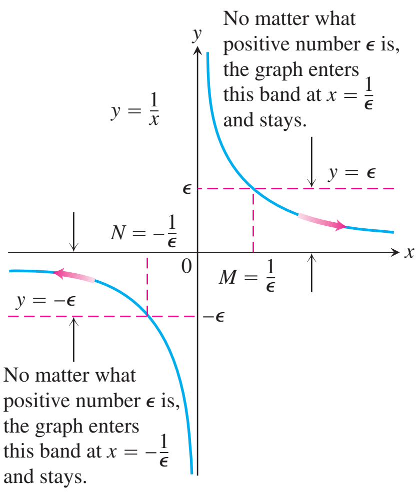


FIGURE 2.32 The geometry behind the argument in Example 6.

The basic facts to be verified by applying the formal definition are

$$\lim_{x \to \pm\infty} k = k \qquad \text{and} \qquad \lim_{x \to \pm\infty} \frac{1}{x} = 0. \tag{3}$$

We prove the latter and leave the former to Exercises 71 and 72.

EXAMPLE 6 Limits at Infinity for $f(x) = \dfrac{1}{x}$

Show that

(a) $\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0$ **(b)** $\displaystyle\lim_{x \to -\infty} \frac{1}{x} = 0.$

Solution

(a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

$$x > M \qquad \Rightarrow \qquad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Figure 2.32). This proves $\lim_{x \to \infty}(1/x) = 0$.

(b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \qquad \Rightarrow \qquad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Figure 2.32). This proves $\lim_{x \to -\infty}(1/x) = 0$. ■

Limits at infinity have properties similar to those of finite limits.

THEOREM 8 **Limit Laws as $x \to \pm\infty$**

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \qquad \text{and} \qquad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\displaystyle\lim_{x \to \pm\infty}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{x \to \pm\infty}(f(x) - g(x)) = L - M$
3. *Product Rule:* $\displaystyle\lim_{x \to \pm\infty}(f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\displaystyle\lim_{x \to \pm\infty}(k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\displaystyle\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then
$$\lim_{x \to \pm\infty}(f(x))^{r/s} = L^{r/s}$$
provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

These properties are just like the properties in Theorem 1, Section 2.2, and we use them the same way.

EXAMPLE 7 Using Theorem 8

$$\text{(a)} \quad \lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \qquad \text{Sum Rule}$$

$$= 5 + 0 = 5 \qquad \text{Known limits}$$

$$\text{(b)} \quad \lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$

$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} \qquad \text{Product rule}$$

$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0 \qquad \text{Known limits}$$

■

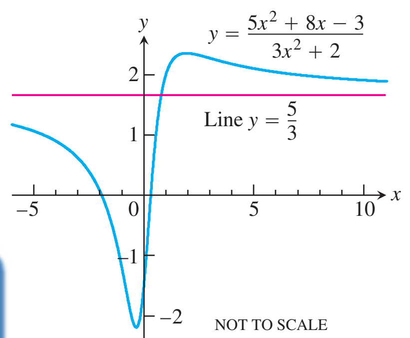


FIGURE 2.33 The graph of the function in Example 8. The graph approaches the line $y = 5/3$ as $|x|$ increases.

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator. What happens then depends on the degrees of the polynomials involved.

EXAMPLE 8 Numerator and Denominator of Same Degree

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)} \qquad \text{Divide numerator and denominator by } x^2.$$

$$= \frac{5+0-0}{3+0} = \frac{5}{3} \qquad \text{See Fig. 2.33.}$$

■

EXAMPLE 9 Degree of Numerator Less Than Degree of Denominator

$$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1} = \lim_{x\to-\infty}\frac{(11/x^2)+(2/x^3)}{2-(1/x^3)} \qquad \text{Divide numerator and denominator by } x^3.$$

$$= \frac{0+0}{2-0} = 0 \qquad \text{See Fig. 2.34.}$$

■

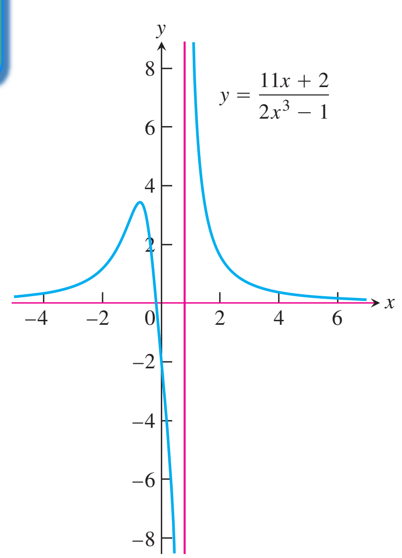


FIGURE 2.34 The graph of the function in Example 9. The graph approaches the x-axis as $|x|$ increases.

We give an example of the case when the degree of the numerator is greater than the degree of the denominator in the next section (Example 8, Section 2.5).

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

Looking at $f(x) = 1/x$ (See Figure 2.31), we observe that the x-axis is an asymptote of the curve on the right because

$$\lim_{x\to\infty}\frac{1}{x} = 0$$

and on the left because

$$\lim_{x\to-\infty}\frac{1}{x} = 0.$$

We say that the x-axis is a *horizontal asymptote* of the graph of $f(x) = 1/x$.

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

The curve

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

sketched in Figure 2.33 (Example 8) has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x\to\infty} f(x) = \frac{5}{3} \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \frac{5}{3}.$$

EXAMPLE 10 Substituting a New Variable

Video Video

Find $\lim_{x\to\infty} \sin(1/x)$.

Solution We introduce the new variable $t = 1/x$. From Example 6, we know that $t \to 0^+$ as $x \to \infty$ (see Figure 2.31). Therefore,

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0^+} \sin t = 0.$$ ■

The Sandwich Theorem Revisited

The Sandwich Theorem also holds for limits as $x \to \pm\infty$.

EXAMPLE 11 A Curve May Cross Its Horizontal Asymptote

Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.35).

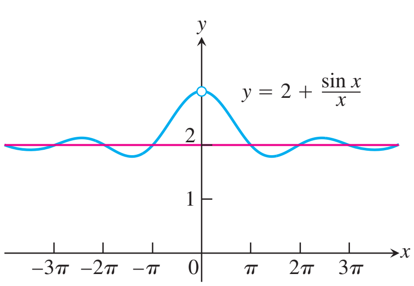


FIGURE 2.35 A curve may cross one of its asymptotes infinitely often (Example 11).

This example illustrates that a curve may cross one of its horizontal asymptotes, perhaps many times.

Oblique Asymptotes

If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an **oblique (slanted) asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$. Here's an example.

EXAMPLE 12 Finding an Oblique Asymptote

Find the oblique asymptote for the graph of

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

in Figure 2.36.

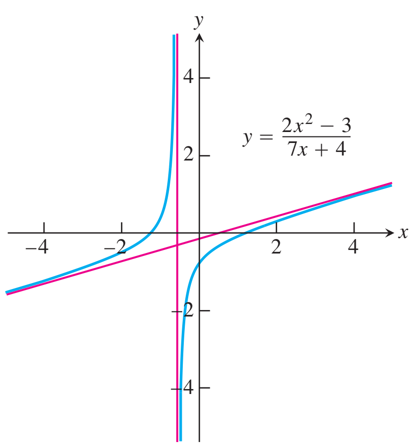


FIGURE 2.36 The function in Example 12 has an oblique asymptote.

Solution By long division, we find

$$\begin{aligned} f(x) &= \frac{2x^2 - 3}{7x + 4} \\ &= \underbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}_{\text{linear function } g(x)} + \underbrace{\frac{-115}{49(7x + 4)}}_{\text{remainder}} \end{aligned}$$

As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the (slanted) line

$$g(x) = \frac{2}{7}x - \frac{8}{49}$$

an asymptote of the graph of f (Figure 2.36). The line $y = g(x)$ is an asymptote both to the right and to the left. In the next section you will see that the function $f(x)$ grows arbitrarily large in absolute value as x approaches $-4/7$, where the denominator becomes zero (Figure 2.36).