In this section we extend the concept of limit to *infinite limits*, which are not limits as before, but rather an entirely new use of the term limit. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large, positive or negative. We continue our analysis of graphs of rational functions from the last section, using vertical asymptotes and dominant terms for numerically large values of $x$.

### Infinite Limits



You can get as high as you want by taking $x$ close enough to 0. No matter how high $B$ is, the graph goes higher.

$y = \frac{1}{x}$

No matter how low $-B$ is, the graph goes lower.

You can get as low as you want by taking $x$ close enough to 0.

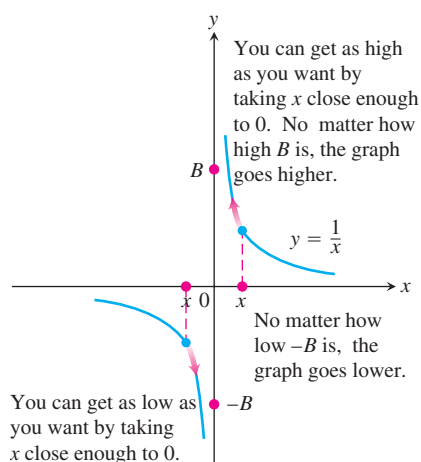**FIGURE 2.37**  One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of $f$ grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number $B$, however large, the values of $f$ become larger still (Figure 2.37). Thus, $f$ has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of $f$ by saying that $f(x)$ approaches $\infty$ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

In writing this, we are *not* saying that the limit exists. Nor are we saying that there is a real number $\infty$, for there is no such number. Rather, we are saying that $\lim_{x \to 0^+} (1/x)$ *does not exist because $1/x$ becomes arbitrarily large and positive as $x \to 0^+$*.

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of $f$ eventually lie below $-B$. (See Figure 2.37.) We write

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative*.

**EXAMPLE 1**    One-Sided Infinite Limits

Find $\lim\limits_{x \to 1^+} \dfrac{1}{x - 1}$    and    $\lim\limits_{x \to 1^-} \dfrac{1}{x - 1}$.
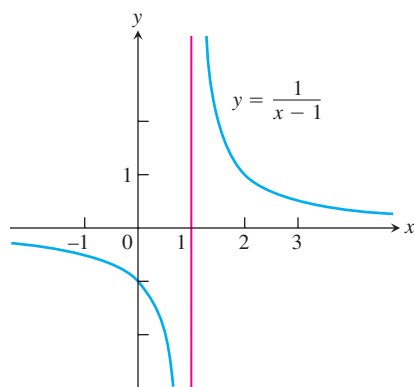


**FIGURE 2.38**  Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 1).
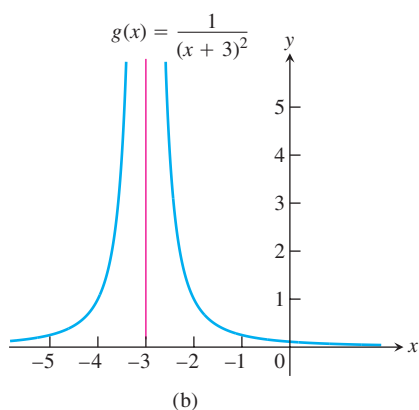
**Geometric Solution**    The graph of $y = 1/(x - 1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.38). Therefore, $y = 1/(x - 1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x \to 1^+} \frac{1}{x - 1} = \infty \qquad \text{and} \qquad \lim_{x \to 1^-} \frac{1}{x - 1} = -\infty.$$

**Analytic Solution**    Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x - 1) \to 0^+$ and $1/(x - 1) \to \infty$. As $x \to 1^-$, we have $(x - 1) \to 0^-$ and $1/(x - 1) \to -\infty$. ∎

No matter how high $B$ is, the graph goes higher.

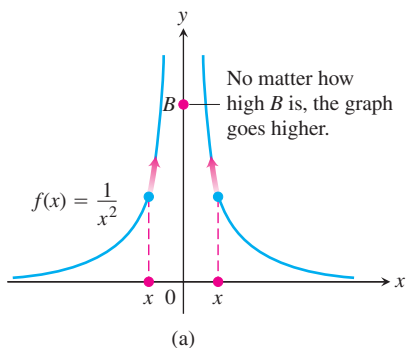$f(x) = \dfrac{1}{x^2}$

(a)



$g(x) = \dfrac{1}{(x + 3)^2}$

(b)

**FIGURE 2.39** The graphs of the functions in Example 2. (a) $f(x)$ approaches infinity as $x \to 0$. (b) $g(x)$ approaches infinity as $x \to -3$.

**EXAMPLE 2**   Two-Sided Infinite Limits

Discuss the behavior of

**(a)** $f(x) = \dfrac{1}{x^2}$ near $x = 0$,

**(b)** $g(x) = \dfrac{1}{(x + 3)^2}$ near $x = -3$.

**Solution**

**(a)** As $x$ approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.39a):

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

**(b)** The graph of $g(x) = 1/(x + 3)^2$ is the graph of $f(x) = 1/x^2$ shifted 3 units to the left (Figure 2.39b). Therefore, $g$ behaves near $-3$ exactly the way $f$ behaves near 0.

$$\lim_{x \to -3} g(x) = \lim_{x \to -3} \frac{1}{(x + 3)^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x \to 0}(1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as $x$ approaches zero from either side, so we can say that $\lim_{x \to 0}(1/x^2) = \infty$.

**EXAMPLE 3**   Rational Functions Can Behave in Various Ways Near Zeros of Their Denominators

**(a)** $\displaystyle\lim_{x \to 2} \frac{(x - 2)^2}{x^2 - 4} = \lim_{x \to 2} \frac{(x - 2)^2}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{x - 2}{x + 2} = 0$

**(b)** $\displaystyle\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{1}{x + 2} = \frac{1}{4}$

**(c)** $\displaystyle\lim_{x \to 2^+} \frac{x - 3}{x^2 - 4} = \lim_{x \to 2^+} \frac{x - 3}{(x - 2)(x + 2)} = -\infty$ 　　The values are negative for $x > 2$, $x$ near 2.

**(d)** $\displaystyle\lim_{x \to 2^-} \frac{x - 3}{x^2 - 4} = \lim_{x \to 2^-} \frac{x - 3}{(x - 2)(x + 2)} = \infty$ 　　The values are positive for $x < 2$, $x$ near 2.

**(e)** $\displaystyle\lim_{x \to 2} \frac{x - 3}{x^2 - 4} = \lim_{x \to 2} \frac{x - 3}{(x - 2)(x + 2)}$ does not exist. 　　See parts (c) and (d).

**(f)** $\displaystyle\lim_{x \to 2} \frac{2 - x}{(x - 2)^3} = \lim_{x \to 2} \frac{-(x - 2)}{(x - 2)^3} = \lim_{x \to 2} \frac{-1}{(x - 2)^2} = -\infty$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero in the denominator.

## Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number $L$ for all $x$ sufficiently close to $x_0$, the definitions of infinite limits require $f(x)$ to lie arbitrarily far from the ori-
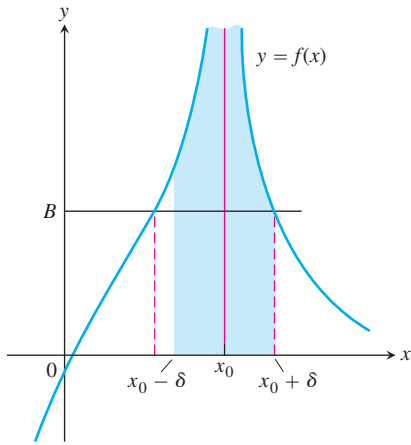
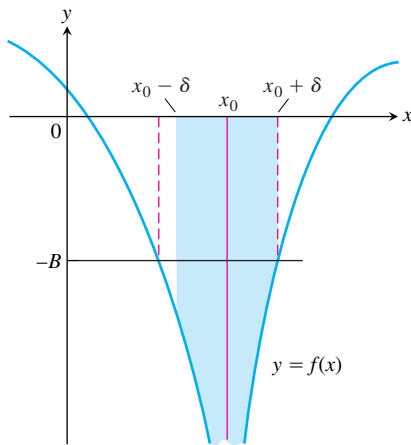**FIGURE 2.40**  $f(x)$ approaches infinity as $x \rightarrow x_0$.



**FIGURE 2.41**  $f(x)$ approaches negative infinity as $x \rightarrow x_0$.

gin. Except for this change, the language is identical with what we have seen before. Figures 2.40 and 2.41 accompany these definitions.

---

**DEFINITIONS**    **Infinity, Negative Infinity as Limits**

1.  We say that **$f(x)$ approaches infinity as $x$ approaches $x_0$**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number $B$ there exists a corresponding $\delta > 0$ such that for all $x$

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2.  We say that **$f(x)$ approaches negative infinity as $x$ approaches $x_0$**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all $x$

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$

---

The precise definitions of one-sided infinite limits at $x_0$ are similar and are stated in the exercises.

**EXAMPLE 4**    Using the Definition of Infinite Limits

Prove that $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

**Solution**    Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if } x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \ge B.$$

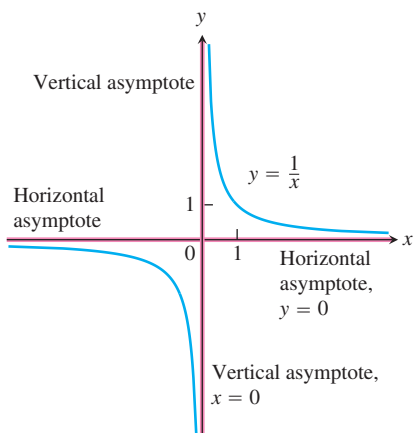Therefore, by definition,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

**FIGURE 2.42** The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

## Vertical Asymptotes
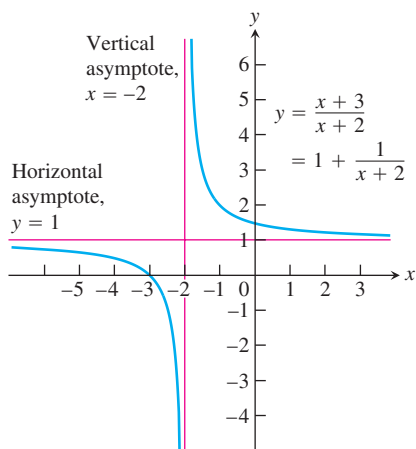
Notice that the distance between a point on the graph of $y = 1/x$ and the $y$-axis approaches zero as the point moves vertically along the graph and away from the origin (Figure 2.42). This behavior occurs because

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

We say that the line $x = 0$ (the $y$-axis) is a *vertical asymptote* of the graph of $y = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

---

**DEFINITION** Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

---

**EXAMPLE 5** Looking for Asymptotes

Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x + 3}{x + 2}.$$

**Solution** We are interested in the behavior as $x \to \pm\infty$ and as $x \to -2$, where the denominator is zero.

The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$.

$$\begin{array}{r} 1 \phantom{00000} \\ x + 2 \overline{)x + 3} \\ \underline{x + 2} \\ 1 \end{array}$$

This result enables us to rewrite $y$:

$$y = 1 + \frac{1}{x + 2}.$$

We now see that the curve in question is the graph of $y = 1/x$ shifted 1 unit up and 2 units left (Figure 2.43). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. ∎



**FIGURE 2.43** The lines $y = 1$ and $x = -2$ are asymptotes of the curve $y = (x + 3)/(x + 2)$ (Example 5).

**EXAMPLE 6** Asymptotes Need Not Be Two-Sided

Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

**Solution** We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that $f$ is an even function of $x$, so its graph is symmetric with respect to the $y$-axis.

**(a)** *The behavior as $x \to \pm\infty$.* Since $\lim_{x \to \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well
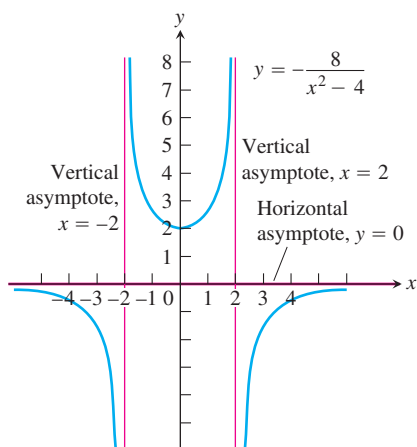
**FIGURE 2.44** Graph of $y = -8/(x^2 - 4)$. Notice that the curve approaches the $x$-axis from only one side. Asymptotes do not have to be two-sided (Example 6).

(Figure 2.44). Notice that the curve approaches the $x$-axis from only the negative side (or from below).

**(b)** *The behavior as $x \to \pm 2$.* Since

$$\lim_{x \to 2^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the same holds for the line $x = -2$.

There are no other asymptotes because $f$ has a finite limit at every other point. ∎

**EXAMPLE 7** Curves with Infinitely Many Asymptotes

The curves

$$y = \sec x = \frac{1}{\cos x} \qquad \text{and} \qquad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Figure 2.45).
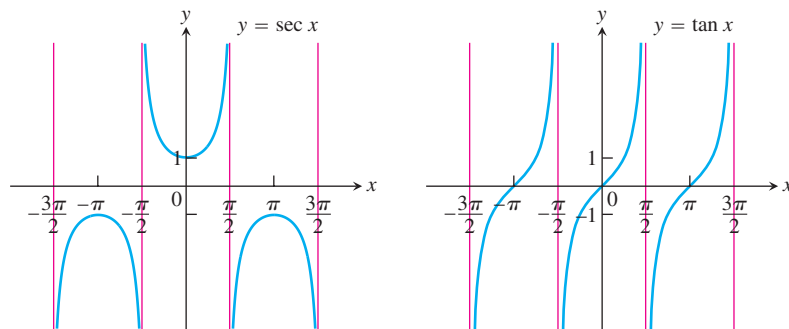


**FIGURE 2.45** The graphs of sec $x$ and tan $x$ have infinitely many vertical asymptotes (Example 7).

The graphs of

$$y = \csc x = \frac{1}{\sin x} \qquad \text{and} \qquad y = \cot x = \frac{\cos x}{\sin x}$$

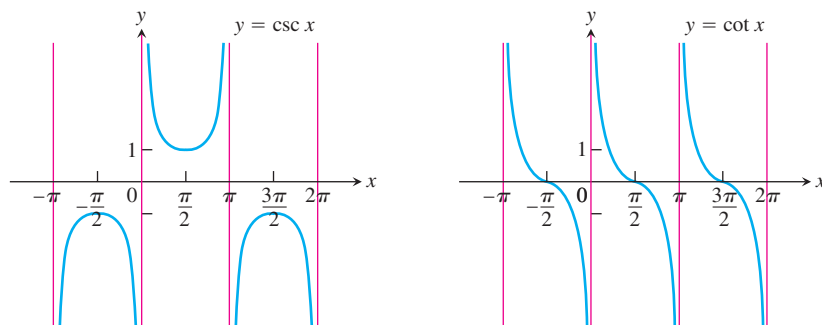have vertical asymptotes at integer multiples of $\pi$, where $\sin x = 0$ (Figure 2.46).



**FIGURE 2.46** The graphs of csc $x$ and cot $x$ (Example 7). ∎

$$y = \frac{x^2 - 3}{2x - 4} = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

The vertical distance between curve and line goes to zero as $x \to \infty$

$x = 2$

Oblique asymptote

$y = \frac{x}{2} + 1$

Vertical asymptote, $x = 2$

**FIGURE 2.47** The graph of $f(x) = (x^2 - 3)/(2x - 4)$ has a vertical asymptote and an oblique asymptote (Example 8).

**EXAMPLE 8**    A Rational Function with Degree of Numerator Greater than Degree of Denominator

Find the asymptotes of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

**Solution**    We are interested in the behavior as $x \to \pm\infty$ and also as $x \to 2$, where the denominator is zero. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{)\, x^2 - 3\phantom{00}} \\ \underline{x^2 - 2x\phantom{000}} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\frac{x}{2} + 1}_{\text{linear}} + \underbrace{\frac{1}{2x - 4}}_{\text{remainder}}.$$

Since $\lim_{x \to 2^+} f(x) = \infty$ and $\lim_{x \to 2^-} f(x) = -\infty$, the line $x = 2$ is a two-sided vertical asymptote. As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to (x/2) + 1$. The line $y = (x/2) + 1$ is an oblique asymptote both to the right and to the left (Figure 2.47). ∎

Notice in Example 8, that if the degree of the numerator in a rational function is greater than the degree of the denominator, then the limit is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator as $|x|$ becomes large.

## Dominant Terms

Of all the observations we can make quickly about the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Example 8, probably the most useful is that

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}.$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{For } x \text{ numerically large}$$

$$f(x) \approx \frac{1}{2x - 4} \qquad \text{For } x \text{ near 2}$$

If we want to know how $f$ behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when $x$ is numerically large and the contribution of $1/(2x - 4)$ to the total
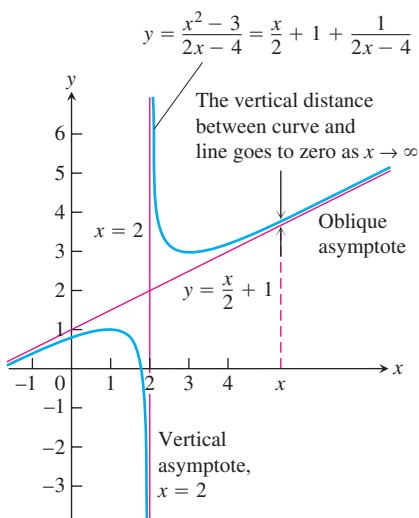
value of $f$ is insignificant. It behaves like $1/(2x - 4)$ when $x$ is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when $x$ is numerically large, and we say that $1/(2x - 4)$ dominates when $x$ is near 2. **Dominant terms** like these are the key to predicting a function's behavior. Here's another example.

### EXAMPLE 9   Two Graphs Appearing Identical on a Large Scale

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although $f$ and $g$ are quite different for numerically small values of $x$, they are virtually identical for $|x|$ very large.

**Solution**   The graphs of $f$ and $g$ behave quite differently near the origin (Figure 2.48a), but appear as virtually identical on a larger scale (Figure 2.48b).
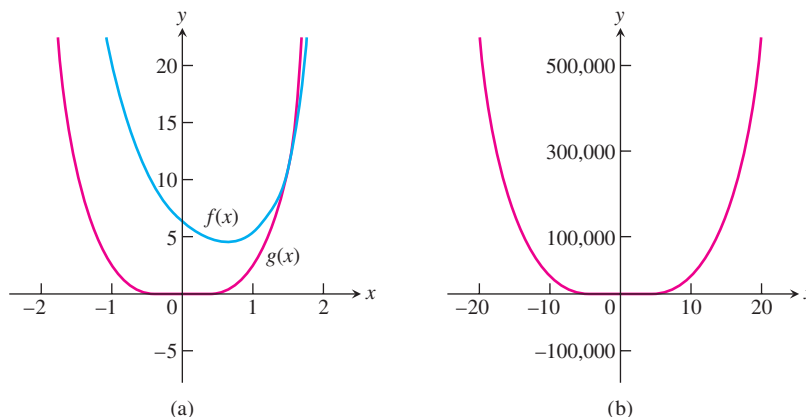


**FIGURE 2.48**   The graphs of $f$ and $g$, (a) are distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 9).

We can test that the term $3x^4$ in $f$, represented graphically by $g$, dominates the polynomial $f$ for numerically large values of $x$ by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \lim_{x \to \pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4}$$

$$= \lim_{x \to \pm\infty} \left( 1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4} \right)$$

$$= 1,$$

so that $f$ and $g$ are nearly identical for $|x|$ large.                                                        ∎