Chapter 2 Practice Exercises

Limits and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0$, and 1. Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \leq -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

3. Suppose that $f(t)$ and $g(t)$ are defined for all t and that $\lim_{t \to t_0} f(t) = -7$ and $\lim_{t \to t_0} g(t) = 0$. Find the limit as $t \to t_0$ of the following functions.

a. $3f(t)$

b. $(f(t))^2$

c. $f(t) \cdot g(t)$

d. $\dfrac{f(t)}{g(t) - 7}$

e. $\cos(g(t))$

f. $|f(t)|$

g. $f(t) + g(t)$

h. $1/f(t)$

4. Suppose that $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to 0} f(x) = 1/2$ and $\lim_{x\to 0} g(x) = \sqrt{2}$. Find the limits as $x \to 0$ of the following functions.

a. $-g(x)$ **b.** $g(x)\cdot f(x)$

c. $f(x) + g(x)$ **d.** $1/f(x)$

e. $x + f(x)$ **f.** $\dfrac{f(x)\cdot \cos x}{x-1}$

In Exercises 5 and 6, find the value that $\lim_{x\to 0} g(x)$ must have if the given limit statements hold.

5. $\lim_{x\to 0}\left(\dfrac{4-g(x)}{x}\right) = 1$ **6.** $\lim_{x\to -4}\left(x \lim_{x\to 0} g(x)\right) = 2$

7. On what intervals are the following functions continuous?

a. $f(x) = x^{1/3}$ **b.** $g(x) = x^{3/4}$

c. $h(x) = x^{-2/3}$ **d.** $k(x) = x^{-1/6}$

8. On what intervals are the following functions continuous?

a. $f(x) = \tan x$ **b.** $g(x) = \csc x$

c. $h(x) = \dfrac{\cos x}{x-\pi}$ **d.** $k(x) = \dfrac{\sin x}{x}$

Finding Limits

In Exercises 9–16, find the limit or explain why it does not exist.

9. $\lim \dfrac{x^2-4x+4}{x^3+5x^2-14x}$

a. as $x \to 0$ **b.** as $x \to 2$

10. $\lim \dfrac{x^2+x}{x^5+2x^4+x^3}$

a. as $x \to 0$ **b.** as $x \to -1$

11. $\lim_{x\to 1} \dfrac{1-\sqrt{x}}{1-x}$ **12.** $\lim_{x\to a} \dfrac{x^2-a^2}{x^4-a^4}$

13. $\lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$ **14.** $\lim_{x\to 0} \dfrac{(x+h)^2-x^2}{h}$

15. $\lim_{x\to 0} \dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}$ **16.** $\lim_{x\to 0} \dfrac{(2+x)^3-8}{x}$

In Exercises 17–20, find the limit of $g(x)$ as x approaches the indicated value.

17. $\lim_{x\to 0^+} (4g(x))^{1/3} = 2$ **18.** $\lim_{x\to\sqrt{5}} \dfrac{1}{x+g(x)} = 2$

19. $\lim_{x\to 1} \dfrac{3x^2+1}{g(x)} = \infty$ **20.** $\lim_{x\to -2} \dfrac{5-x^2}{\sqrt{g(x)}} = 0$

Limits at Infinity

Find the limits in Exercises 21–30.

21. $\lim_{x\to\infty} \dfrac{2x+3}{5x+7}$ **22.** $\lim_{x\to -\infty} \dfrac{2x^2+3}{5x^2+7}$

23. $\lim_{x\to -\infty} \dfrac{x^2-4x+8}{3x^3}$ **24.** $\lim_{x\to\infty} \dfrac{1}{x^2-7x+1}$

25. $\lim_{x\to -\infty} \dfrac{x^2-7x}{x+1}$ **26.** $\lim_{x\to\infty} \dfrac{x^4+x^3}{12x^3+128}$

27. $\lim_{x\to\infty} \dfrac{\sin x}{\lfloor x \rfloor}$ (If you have a grapher, try graphing the function for $-5 \le x \le 5$.)

28. $\lim_{\theta\to\infty} \dfrac{\cos\theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to "see" the limit at infinity.)

29. $\lim_{x\to\infty} \dfrac{x+\sin x+2\sqrt{x}}{x+\sin x}$ **30.** $\lim_{x\to\infty} \dfrac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x}$

Continuous Extension

31. Can $f(x) = x(x^2-1)/|x^2-1|$ be extended to be continuous at $x = 1$ or -1? Give reasons for your answers. (Graph the function—you will find the graph interesting.)

32. Explain why the function $f(x) = \sin(1/x)$ has no continuous extension to $x = 0$.

T In Exercises 33–36, graph the function to see whether it appears to have a continuous extension to the given point a. If it does, use Trace and Zoom to find a good candidate for the extended function's value at a. If the function does not appear to have a continuous extension, can it be extended to be continuous from the right or left? If so, what do you think the extended function's value should be?

33. $f(x) = \dfrac{x-1}{x-\sqrt[4]{x}}$, $a = 1$ **34.** $g(\theta) = \dfrac{5\cos\theta}{4\theta-2\pi}$, $a = \pi/2$

35. $h(t) = (1+|t|)^{1/t}$, $a = 0$ **36.** $k(x) = \dfrac{x}{1-2^{|x|}}$, $a = 0$

Roots

T 37. Let $f(x) = x^3 - x - 1$.

a. Show that f has a zero between -1 and 2.

b. Solve the equation $f(x) = 0$ graphically with an error of magnitude at most 10^{-8}.

c. It can be shown that the exact value of the solution in part (b) is

$$\left(\frac{1}{2}+\frac{\sqrt{69}}{18}\right)^{1/3} + \left(\frac{1}{2}-\frac{\sqrt{69}}{18}\right)^{1/3}$$

Evaluate this exact answer and compare it with the value you found in part (b).

T 38. Let $f(\theta) = \theta^3 - 2\theta + 2$.

a. Show that f has a zero between -2 and 0.

b. Solve the equation $f(\theta) = 0$ graphically with an error of magnitude at most 10^{-4}.

c. It can be shown that the exact value of the solution in part (b) is

$$\left(\sqrt{\frac{19}{27}}-1\right)^{1/3} - \left(\sqrt{\frac{19}{27}}+1\right)^{1/3}$$

Evaluate this exact answer and compare it with the value you found in part (b).