3.4 Derivatives of Trigonometric Functions

Many of the phenomena we want information about are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\ &= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Sine angle sum identity} \\ &= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\ &= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 && \text{Example 5(a) and Theorem 7, Section 2.4} \\ &= \cos x. \end{aligned}$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 Derivatives Involving the Sine

(a) $y = x^2 - \sin x$:

$$\begin{aligned} \frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule} \\ &= 2x - \cos x. \end{aligned}$$

(b) $y = x^2 \sin x$:

$$\begin{aligned} \frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \sin x && \text{Product Rule} \\ &= x^2 \cos x + 2x \sin x. \end{aligned}$$

(c) $y = \dfrac{\sin x}{x}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} && \text{Quotient Rule} \\ &= \frac{x \cos x - \sin x}{x^2}. \end{aligned}$$

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Derivative of the Cosine Function

With the help of the angle sum formula for the cosine,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

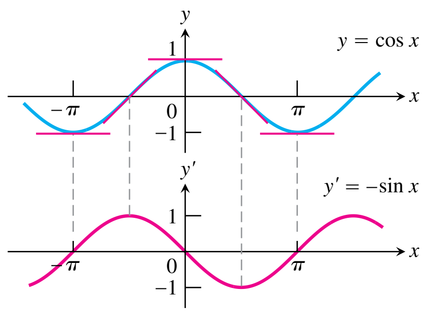

FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

we have

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} && \text{Derivative definition}\\
&= \lim_{h\to 0}\frac{(\cos x\cos h - \sin x\sin h)-\cos x}{h} && \text{Cosine angle sum identity}\\
&= \lim_{h\to 0}\frac{\cos x(\cos h - 1)-\sin x\sin h}{h}\\
&= \lim_{h\to 0}\cos x\cdot\frac{\cos h - 1}{h} - \lim_{h\to 0}\sin x\cdot\frac{\sin h}{h}\\
&= \cos x\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\cdot\lim_{h\to 0}\frac{\sin h}{h}\\
&= \cos x\cdot 0 - \sin x\cdot 1 && \text{Example 5(a) and Theorem 7, Section 2.4}\\
&= -\sin x.
\end{aligned}$$

> **The derivative of the cosine function is the negative of the sine function:**
>
> $$\frac{d}{dx}(\cos x) = -\sin x$$

Figure 3.23 shows a way to visualize this result.

EXAMPLE 2 Derivatives Involving the Cosine

You Try It

(a) $y = 5x + \cos x$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) && \text{Sum Rule}\\
&= 5 - \sin x.
\end{aligned}$$

Video

(b) $y = \sin x\cos x$:

$$\begin{aligned}
\frac{dy}{dx} &= \sin x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(\sin x) && \text{Product Rule}\\
&= \sin x(-\sin x) + \cos x(\cos x)\\
&= \cos^2 x - \sin^2 x.
\end{aligned}$$

Video

(c) $y = \dfrac{\cos x}{1-\sin x}$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} && \text{Quotient Rule}\\
&= \frac{(1-\sin x)(-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2}\\
&= \frac{1-\sin x}{(1-\sin x)^2} && \sin^2 x + \cos^2 x = 1\\
&= \frac{1}{1-\sin x}.
\end{aligned}$$

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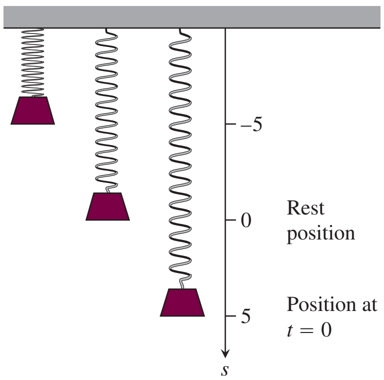

FIGURE 3.24 A body hanging from a vertical spring and then displaced oscillates above and below its rest position. Its motion is described by trigonometric functions (Example 3).

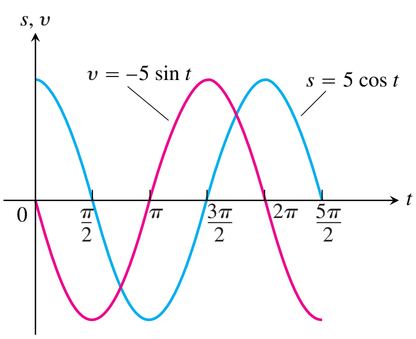

FIGURE 3.25 The graphs of the position and velocity of the body in Example 3.

Simple Harmonic Motion

The motion of a body bobbing freely up and down on the end of a spring or bungee cord is an example of *simple harmonic motion*. The next example describes a case in which there are no opposing forces such as friction or buoyancy to slow the motion down.

EXAMPLE 3 Motion on a Spring

A body hanging from a spring (Figure 3.24) is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t?

Solution We have

Position: $s = 5 \cos t$

Velocity: $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(5 \cos t) = -5 \sin t$

Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(-5 \sin t) = -5 \cos t.$

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π.
2. The velocity $v = -5 \sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs show in Figure 3.25. Hence, the speed of the weight, $|v| = 5|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5 \cos t = \pm 5$, at the endpoints of the interval of motion.
3. The acceleration value is always the exact opposite of the position value. When the weight is above the rest position, gravity is pulling it back down; when the weight is below the rest position, the spring is pulling it back up.
4. The acceleration, $a = -5 \cos t$, is zero only at the rest position, where $\cos t = 0$ and the force of gravity and the force from the spring offset each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. ■

EXAMPLE 4 Jerk

The jerk of the simple harmonic motion in Example 3 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5 \cos t) = 5 \sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign. ■

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

To show a typical calculation, we derive the derivative of the tangent function. The other derivations are left to Exercise 50.

EXAMPLE 5

Find $d(\tan x)/dx$.

Solution

$$\begin{aligned}\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule}\\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

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EXAMPLE 6

Find y'' if $y = \sec x$.

Solution

$$\begin{aligned} y &= \sec x\\ y' &= \sec x \tan x\\ y'' &= \frac{d}{dx}(\sec x \tan x)\\ &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Product Rule}\\ &= \sec x(\sec^2 x) + \tan x(\sec x \tan x)\\ &= \sec^3 x + \sec x \tan^2 x\end{aligned}$$

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The differentiability of the trigonometric functions throughout their domains gives another proof of their continuity at every point in their domains (Theorem 1, Section 3.1). So we can calculate limits of algebraic combinations and composites of trigonometric functions by direct substitution.

EXAMPLE 7 Finding a Trigonometric Limit

$$\lim_{x \to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$ ■