EXERCISES 3.5

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = -x/3$
5. $y = \cos u, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = 10x - 5$
8. $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{x}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \sin^3 x$
18. $y = 5\cos^{-4} x$

Find the derivatives of the functions in Exercises 19–38.

19. $p = \sqrt{3 - t}$
20. $q = \sqrt{2r - r^2}$
21. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$
22. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$
23. $r = (\csc\theta + \cot\theta)^{-1}$
24. $r = -(\sec\theta + \tan\theta)^{-1}$
25. $y = x^2\sin^4 x + x\cos^{-2} x$
26. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$
27. $y = \frac{1}{21}(3x - 2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1}$
28. $y = (5 - 2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$
29. $y = (4x + 3)^4(x + 1)^{-3}$
30. $y = (2x - 5)^{-1}(x^2 - 5x)^6$
31. $h(x) = x\tan\left(2\sqrt{x}\right) + 7$
32. $k(x) = x^2\sec\left(\frac{1}{x}\right)$
33. $f(\theta) = \left(\frac{\sin\theta}{1 + \cos\theta}\right)^2$
34. $g(t) = \left(\frac{1 + \cos t}{\sin t}\right)^{-1}$
35. $r = \sin(\theta^2)\cos(2\theta)$
36. $r = \sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$
37. $q = \sin\left(\frac{t}{\sqrt{t + 1}}\right)$
38. $q = \cot\left(\frac{\sin t}{t}\right)$

In Exercises 39–48, find dy/dt.

39. $y = \sin^2(\pi t - 2)$
40. $y = \sec^2 \pi t$
41. $y = (1 + \cos 2t)^{-4}$
42. $y = (1 + \cot(t/2))^{-2}$

43. $y = \sin(\cos(2t - 5))$

44. $y = \cos\left(5 \sin\left(\frac{t}{3}\right)\right)$

45. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$

46. $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$

47. $y = \sqrt{1 + \cos(t^2)}$

48. $y = 4 \sin\left(\sqrt{1 + \sqrt{t}}\right)$

Second Derivatives

Find y'' in Exercises 49–52.

49. $y = \left(1 + \frac{1}{x}\right)^3$

50. $y = \left(1 - \sqrt{x}\right)^{-1}$

51. $y = \frac{1}{9}\cot(3x - 1)$

52. $y = 9 \tan\left(\frac{x}{3}\right)$

Finding Numerical Values of Derivatives

In Exercises 53–58, find the value of $(f \circ g)'$ at the given value of x.

53. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

54. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$

55. $f(u) = \cot\frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

56. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

57. $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

58. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

59. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $2f(x), \quad x = 2$
b. $f(x) + g(x), \quad x = 3$
c. $f(x) \cdot g(x), \quad x = 3$
d. $f(x)/g(x), \quad x = 2$
e. $f(g(x)), \quad x = 2$
f. $\sqrt{f(x)}, \quad x = 2$
g. $1/g^2(x), \quad x = 3$
h. $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

60. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x,

a. $5f(x) - g(x), \quad x = 1$
b. $f(x)g^3(x), \quad x = 0$
c. $\frac{f(x)}{g(x) + 1}, \quad x = 1$
d. $f(g(x)), \quad x = 0$
e. $g(f(x)), \quad x = 0$
f. $(x^{11} + f(x))^{-2}, \quad x = 1$
g. $f(x + g(x)), \quad x = 0$

61. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

62. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Choices in Composition

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 63 and 64.

63. Find dy/dx if $y = x$ by using the Chain Rule with y as a composite of

a. $y = (u/5) + 7$ and $u = 5x - 35$

b. $y = 1 + (1/u)$ and $u = 1/(x - 1)$.

64. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composite of

a. $y = u^3$ and $u = \sqrt{x}$

b. $y = \sqrt{u}$ and $u = x^3$.

Tangents and Slopes

65. a. Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.

b. Slopes on a tangent curve What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

66. Slopes on sine curves

a. Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.

b. Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.

c. For a given m, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.

d. The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

Finding Cartesian Equations from Parametric Equations

Exercises 67–78 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

67. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi$

68. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

69. $x = 4\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$

70. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

71. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

72. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

73. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$

74. $x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$

75. $x = t, \quad y = \sqrt{1 - t^2}, \quad -1 \le t \le 0$

76. $x = \sqrt{t + 1}, \quad y = \sqrt{t}, \quad t \ge 0$

77. $x = \sec^2 t - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

78. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

Determining Parametric Equations

79. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$

a. once clockwise. **b.** once counterclockwise.

c. twice clockwise. **d.** twice counterclockwise.

(There are many ways to do these, so your answers may not be the same as the ones in the back of the book.)

80. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $(x^2/a^2) + (y^2/b^2) = 1$

a. once clockwise. **b.** once counterclockwise.

c. twice clockwise. **d.** twice counterclockwise.

(As in Exercise 79, there are many correct answers.)

In Exercises 81–86, find a parametrization for the curve.

81. the line segment with endpoints $(-1, -3)$ and $(4, 1)$

82. the line segment with endpoints $(-1, 3)$ and $(3, -2)$

83. the lower half of the parabola $x - 1 = y^2$

84. the left half of the parabola $y = x^2 + 2x$

85. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$

86. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$

Tangents to Parametrized Curves

In Exercises 87–94, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

87. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$

88. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$

89. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$

90. $x = -\sqrt{t + 1}, \quad y = \sqrt{3t}, \quad t = 3$

91. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$

92. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$

93. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$

94. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$

Theory, Examples, and Applications

95. Running machinery too fast Suppose that a piston is moving straight up and down and that its position at time t sec is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

96. Temperatures in Fairbanks, Alaska The graph in Figure 3.33 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

a. On what day is the temperature increasing the fastest?

b. About how many degrees per day is the temperature increasing when it is increasing at its fastest?

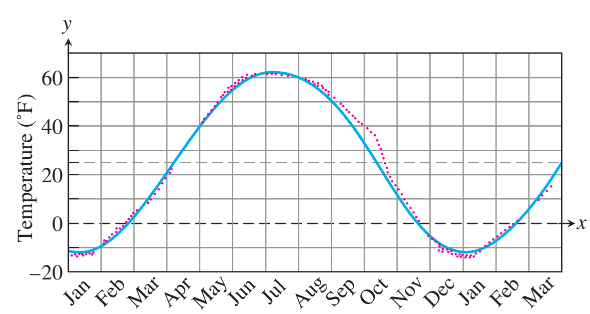


FIGURE 3.33 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points, and the approximating sine function (Exercise 96).

97. Particle motion The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

98. Constant acceleration Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s m from its starting point. Show that the body's acceleration is constant.

99. Falling meteorite The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s km from Earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

100. Particle acceleration A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

101. Temperature and the period of a pendulum For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

102. Chain Rule Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

103. Tangents Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

104. Suppose that $u = g(x)$ is differentiable at $x = -5$, $y = f(u)$ is differentiable at $u = g(-5)$, and $(f \circ g)'(-5)$ is negative. What, if anything, can be said about the values of $g'(-5)$ and $f'(g(-5))$?

T **105. The derivative of sin 2*x*** Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x+h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

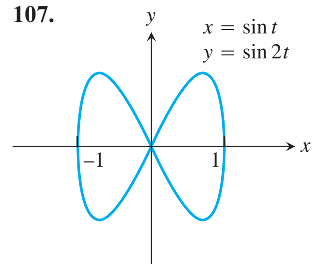
T **106. The derivative of cos (x^2)** Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos((x+h)^2) - \cos(x^2)}{h}$$

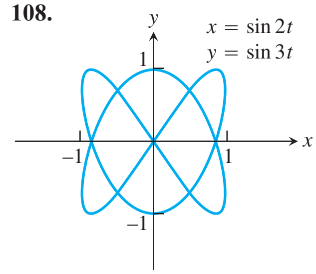
for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

T The curves in Exercises 107 and 108 are called *Bowditch curves* or *Lissajous figures*. In each case, find the point in the interior of the first quadrant where the tangent to the curve is horizontal, and find the equations of the two tangents at the origin.

107.



108.



Using the Chain Rule, show that the power rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 109 and 110.

109. $x^{1/4} = \sqrt{\sqrt{x}}$ **110.** $x^{3/4} = \sqrt{x\sqrt{x}}$

COMPUTER EXPLORATIONS

Trigonometric Polynomials

111. As Figure 3.34 shows, the trigonometric "polynomial"

$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t$$
$$-0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

a. Graph dg/dt (where defined) over $[-\pi, \pi]$.

b. Find df/dt.

c. Graph df/dt. Where does the approximation of dg/dt by df/dt seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.

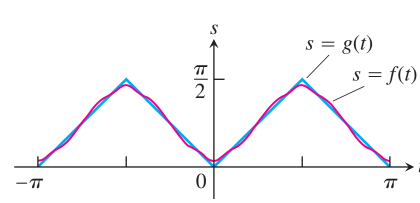


FIGURE 3.34 The approximation of a sawtooth function by a trigonometric "polynomial" (Exercise 111).

112. (*Continuation of Exercise 111.*) In Exercise 111, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the "polynomial"

$$s = h(t) = 1.2732 \sin 2t + 0.4244 \sin 6t + 0.25465 \sin 10t + 0.18189 \sin 14t + 0.14147 \sin 18t$$

graphed in Figure 3.35 approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k.

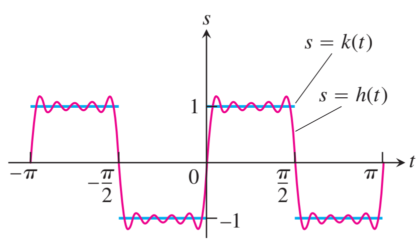


FIGURE 3.35 The approximation of a step function by a trigonometric "polynomial" (Exercise 112).

a. Graph dk/dt (where defined) over $[-\pi, \pi]$.

b. Find dh/dt.

c. Graph dh/dt to see how badly the graph fits the graph of dk/dt. Comment on what you see.

Parametrized Curves

Use a CAS to perform the following steps on the parametrized curves in Exercises 113–116.

a. Plot the curve for the given interval of t values.

b. Find dy/dx and d^2y/dx^2 at the point t_0.

c. Find an equation for the tangent line to the curve at the point defined by the given value t_0. Plot the curve together with the tangent line on a single graph.

113. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad 0 \le t \le 1, \quad t_0 = 1/2$

114. $x = 2t^3 - 16t^2 + 25t + 5, \quad y = t^2 + t - 3, \quad 0 \le t \le 6, \quad t_0 = 3/2$

115. $x = t - \cos t, \quad y = 1 + \sin t, \quad -\pi \le t \le \pi, \quad t_0 = \pi/4$

116. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi, \quad t_0 = \pi/2$