

4.3

Monotonic Functions and The First Derivative Test

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section defines precisely what it means for a function to be increasing or decreasing over an interval, and gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function for the presence of local extreme values.

Increasing Functions and Decreasing Functions

What kinds of functions have positive derivatives or negative derivatives? The answer, provided by the Mean Value Theorem's third corollary, is this: The only functions with positive derivatives are increasing functions; the only functions with negative derivatives are decreasing functions.

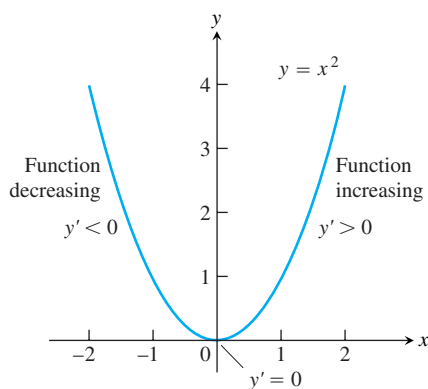


FIGURE 4.21 The function $f(x) = x^2$ is monotonic on the intervals $(-\infty, 0]$ and $[0, \infty)$, but it is not monotonic on $(-\infty, \infty)$.

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I .

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I .
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I .

A function that is increasing or decreasing on I is called **monotonic** on I .

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because of the inequality $<$ comparing the function values, and not \leq , some books say that f is *strictly* increasing or decreasing on I . The interval I may be finite or infinite.

The function $f(x) = x^2$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$ as can be seen from its graph (Figure 4.21). The function f is monotonic on $(-\infty, 0]$ and $[0, \infty)$, but it is not monotonic on $(-\infty, \infty)$. Notice that on the interval $(-\infty, 0)$ the tangents have negative slopes, so the first derivative is always negative there; for $(0, \infty)$ the tangents have positive slopes and the first derivative is positive. The following result confirms these observations.

COROLLARY 3 First Derivative Test for Monotonic Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2 . The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b) and $f(x_2) < f(x_1)$ if f' is negative on (a, b) . ■

Here is how to apply the First Derivative Test to find where a function is increasing and decreasing. If $a < b$ are two critical points for a function f , and if f' exists but is not zero on the interval (a, b) , then f' must be positive on (a, b) or negative there (Theorem 2, Section 3.1). One way we can determine the sign of f' on the interval is simply by evaluating f' for some point x in (a, b) . Then we apply Corollary 3.

EXAMPLE 1 Using the First Derivative Test for Monotonic Functions

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

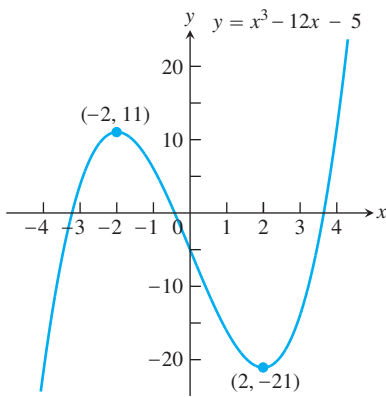


FIGURE 4.22 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f into intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. The behavior of f is determined by then applying Corollary 3 to each subinterval. The results are summarized in the following table, and the graph of f is given in Figure 4.22.

Intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' Evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	+	-	+
Behavior of f	increasing	decreasing	increasing

Corollary 3 is valid for infinite as well as finite intervals, and we used that fact in our analysis in Example 1.

Knowing where a function increases and decreases also tells us how to test for the nature of local extreme values.

HISTORICAL BIOGRAPHY

Edmund Halley
(1656–1742)

First Derivative Test for Local Extrema

In Figure 4.23, at the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.

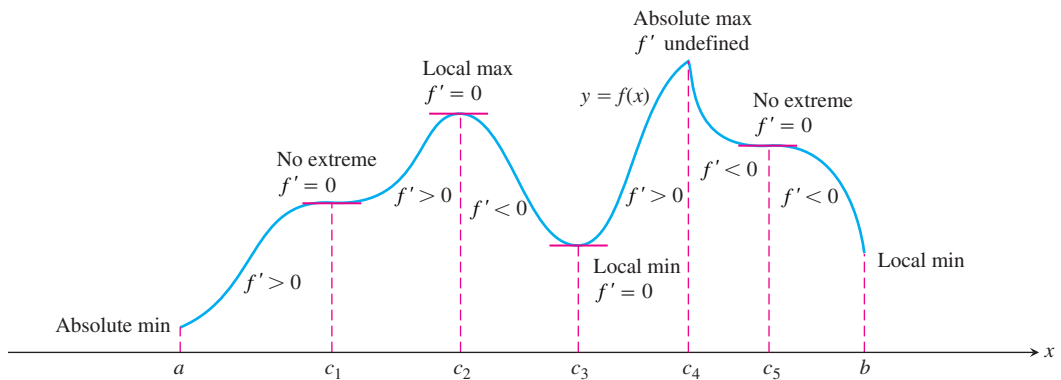


FIGURE 4.23 A function's first derivative tells how the graph rises and falls.

These observations lead to a test for the presence and nature of local extreme values of differentiable functions.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f , and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c , then f has a local minimum at c ;
2. if f' changes from positive to negative at c , then f has a local maximum at c ;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c .

The test for local extrema at endpoints is similar, but there is only one side to consider.

Proof Part (1). Since the sign of f' changes from negative to positive at c , these are numbers a and b such that $f' < 0$ on (a, c) and $f' > 0$ on (c, b) . If $x \in (a, c)$, then $f(c) < f(x)$ because $f' < 0$ implies that f is decreasing on $[a, c]$. If $x \in (c, b)$, then $f(c) < f(x)$ because $f' > 0$ implies that f is increasing on $[c, b]$. Therefore, $f(x) \geq f(c)$ for every $x \in (a, b)$. By definition, f has a local minimum at c .

Parts (2) and (3) are proved similarly. ■

EXAMPLE 2 Using the First Derivative Test for Local Extrema

Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^{4/3} - 4x^{1/3}) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} \\ &= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}} \end{aligned}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x -axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points. We can display the information in a table like the following:

Intervals	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	–	–	+
Behavior of f	decreasing	decreasing	increasing

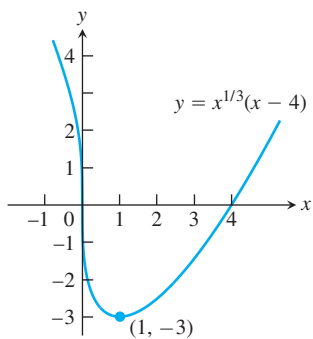


FIGURE 4.24 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

Corollary 3 to the Mean Value Theorem tells us that f decreases on $(-\infty, 0)$, decreases on $(0, 1)$, and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum because the function's values fall toward it from the left and rise away from it on the right. Figure 4.24 shows this value in relation to the function's graph.

Note that $\lim_{x \rightarrow 0} f'(x) = -\infty$, so the graph of f has a vertical tangent at the origin. ■