

4.4

Concavity and Curve Sketching

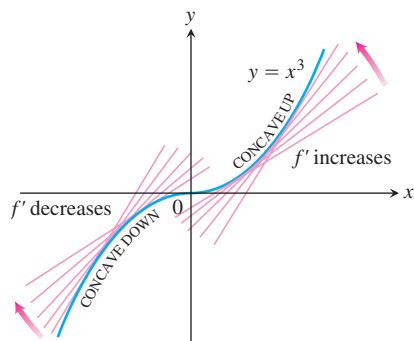


FIGURE 4.25 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

In Section 4.3 we saw how the first derivative tells us where a function is increasing and where it is decreasing. At a critical point of a differentiable function, the First Derivative Test tells us whether there is a local maximum or a local minimum, or whether the graph just continues to rise or fall there.

In this section we see how the second derivative gives information about the way the graph of a differentiable function bends or turns. This additional information enables us to capture key aspects of the behavior of a function and its graph, and then present these features in a sketch of the graph.

Concavity

As you can see in Figure 4.25, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

- (a) **concave up** on an open interval I if f' is increasing on I
- (b) **concave down** on an open interval I if f' is decreasing on I .

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to conclude that f' increases if $f'' > 0$ on I , and decreases if $f'' < 0$.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I .

1. If $f'' > 0$ on I , the graph of f over I is concave up.
2. If $f'' < 0$ on I , the graph of f over I is concave down.

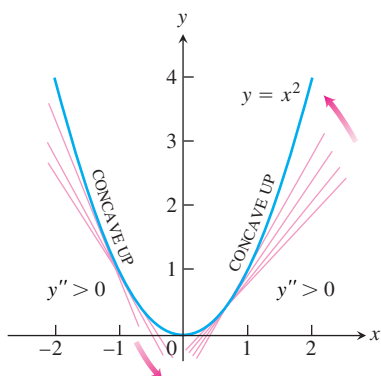


FIGURE 4.26 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

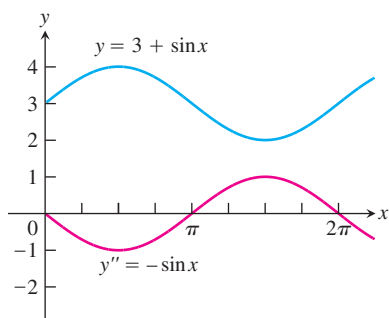


FIGURE 4.27 Using the graph of y'' to determine the concavity of y (Example 2).

If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

EXAMPLE 1 Applying the Concavity Test

- (a) The curve $y = x^3$ (Figure 4.25) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.
- (b) The curve $y = x^2$ (Figure 4.26) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE 2 Determining Concavity

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.27). ■

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

A point on a curve where y'' is positive on one side and negative on the other is a point of inflection. At such a point, y'' is either zero (because derivatives have the Intermediate Value Property) or undefined. If y is a twice-differentiable function, $y'' = 0$ at a point of inflection and y' has a local maximum or minimum.

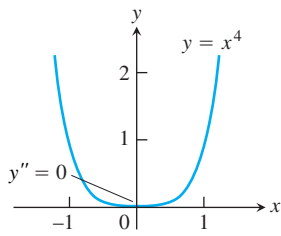


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 3).

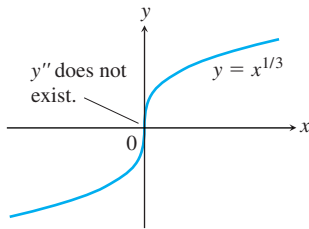


FIGURE 4.29 A point where y'' fails to exist can be a point of inflection (Example 4).

EXAMPLE 3 An Inflection Point May Not Exist Where $y'' = 0$

The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.28). Even though $y'' = 12x^2$ is zero there, it does not change sign. ■

EXAMPLE 4 An Inflection Point May Occur Where y'' Does Not Exist

The curve $y = x^{1/3}$ has a point of inflection at $x = 0$ (Figure 4.29), but y'' does not exist there.

$$y'' = \frac{d^2}{dx^2} (x^{1/3}) = \frac{d}{dx} \left(\frac{1}{3} x^{-2/3} \right) = -\frac{2}{9} x^{-5/3}.$$

We see from Example 3 that a zero second derivative does not always produce a point of inflection. From Example 4, we see that inflection points can also occur where there is no second derivative.

To study the motion of a body moving along a line as a function of time, we often are interested in knowing when the body's acceleration, given by the second derivative, is positive or negative. The points of inflection on the graph of the body's position function reveal where the acceleration changes sign.

EXAMPLE 5 Studying Motion Along a Line

A particle is moving along a horizontal line with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero when $t = 1$ and $t = 11/3$.

Intervals	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	+	-	+
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest), at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Intervals	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	-	+
Graph of s	concave down	concave up

The accelerating force is directed toward the left during the time interval $[0, 7/3]$, is momentarily zero at $t = 7/3$, and is directed toward the right thereafter. ■

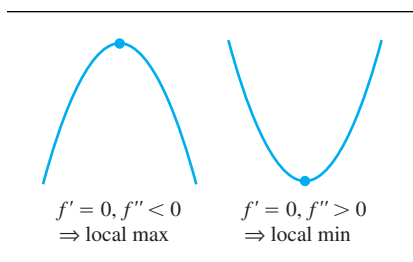
Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and character of local extrema.

THEOREM 5 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



Proof Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c , since f'' is continuous. Therefore, f' is decreasing on I . Since $f'(c) = 0$, the sign of f' changes from positive to negative at c so f has a local maximum at c by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4$, $y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there, $y = -x^4$ has a local maximum, and $y = x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus the test fails. ■

This test requires us to know f'' *only at c itself* and not in an interval about c . This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $f'' = 0$ or if f'' does not exist at $x = c$. When this happens, use the First Derivative Test for local extreme values.

Together f' and f'' tell us the shape of the function's graph, that is, where the critical points are located and what happens at a critical point, where the function is increasing and where it is decreasing, and how the curve is turning or bending as defined by its concavity. We use this information to sketch a graph of the function that captures its key features.

EXAMPLE 6 Using f' and f'' to Graph f

Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

- (a) Identify where the extrema of f occur.
- (b) Find the intervals on which f is increasing and the intervals on which f is decreasing.
- (c) Find where the graph of f is concave up and where it is concave down.
- (d) Sketch the general shape of the graph for f .

- (e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f' . Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	-	-	+
Behavior of f	decreasing	decreasing	increasing

- (a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.
- (b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.
- (c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

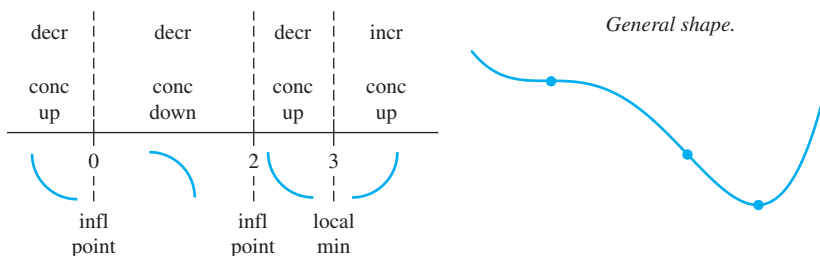
Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	+	-	+
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

- (d) Summarizing the information in the two tables above, we obtain

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

The general shape of the curve is



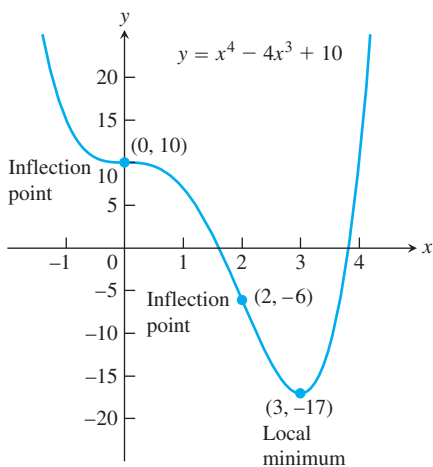


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 6).

- (e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.30 shows the graph of f . ■

The steps in Example 6 help in giving a procedure for graphing to capture the key features of a function and its graph.

Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y'' .
3. Find the critical points of f , and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

EXAMPLE 7 Using the Graphing Strategy

Sketch the graph of $f(x) = \frac{(x + 1)^2}{1 + x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.4).
2. Find f' and f'' .

$$f(x) = \frac{(x + 1)^2}{1 + x^2} \quad \begin{array}{l} \text{x-intercept at } x = -1, \\ \text{y-intercept (} y = 1 \text{) at } \\ x = 0 \end{array}$$

$$f'(x) = \frac{(1 + x^2) \cdot 2(x + 1) - (x + 1)^2 \cdot 2x}{(1 + x^2)^2}$$

$$= \frac{2(1 - x^2)}{(1 + x^2)^2} \quad \begin{array}{l} \text{Critical points:} \\ x = -1, x = 1 \end{array}$$

$$f''(x) = \frac{(1 + x^2)^2 \cdot 2(-2x) - 2(1 - x^2)[2(1 + x^2) \cdot 2x]}{(1 + x^2)^4}$$

$$= \frac{4x(x^2 - 3)}{(1 + x^2)^3} \quad \text{After some algebra}$$

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f . At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the Second Derivative Test. We will see in Step 6 that both are absolute extrema as well.

4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points.* Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.
6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$\begin{aligned} f(x) &= \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} && \text{Expanding numerator} \\ &= \frac{1+(2/x)+(1/x^2)}{(1/x^2)+1}. && \text{Dividing by } x^2 \end{aligned}$$

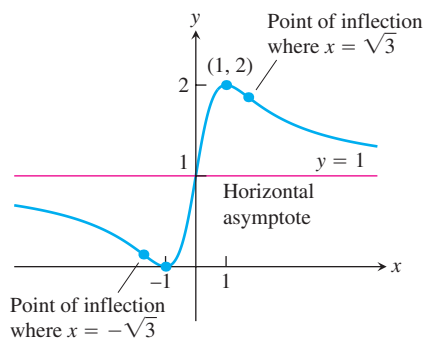


FIGURE 4.31 The graph of $y = \frac{(x+1)^2}{1+x^2}$ (Example 7).

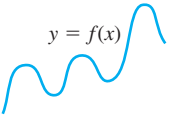
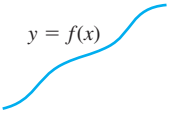
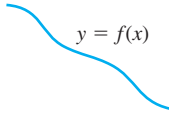
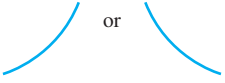
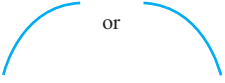

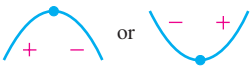


We see that $f(x) \rightarrow 1^+$ as $x \rightarrow \infty$ and that $f(x) \rightarrow 1^-$ as $x \rightarrow -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \leq y \leq 2$).

7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \rightarrow -\infty$, and concave up in its approach to $y = 1$ as $x \rightarrow \infty$. ■

Learning About Functions from Derivatives

As we saw in Examples 6 and 7, we can learn almost everything we need to know about a twice-differentiable function $y = f(x)$ by examining its first derivative. We can find where the function's graph rises and falls and where any local extrema are assumed. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. Information we cannot get from the derivative is how to place the graph in the xy -plane. But, as we discovered in Section 4.2, the only additional information we need to position the graph is the value of f at one point. The derivative does not give us information about the asymptotes, which are found using limits (Sections 2.4 and 2.5).

 <p>$y = f(x)$</p> <p>Differentiable \Rightarrow smooth, connected; graph may rise and fall</p>	 <p>$y = f(x)$</p> <p>$y' > 0 \Rightarrow$ rises from left to right; may be wavy</p>	 <p>$y = f(x)$</p> <p>$y' < 0 \Rightarrow$ falls from left to right; may be wavy</p>
 <p>or</p> <p>$y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall</p>	 <p>or</p> <p>$y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall</p>	 <p>y'' changes sign Inflection point</p>
 <p>or</p> <p>y' changes sign \Rightarrow graph has local maximum or local minimum</p>	 <p>$y' = 0$ and $y'' < 0$ at a point; graph has local maximum</p>	 <p>$y' = 0$ and $y'' > 0$ at a point; graph has local minimum</p>