EXERCISES 4.6

Finding Limits

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to 2} \frac{x-2}{x^2-4}$

2. $\lim_{x\to 0} \frac{\sin 5x}{x}$

3. $\lim_{x\to\infty} \frac{5x^2-3x}{7x^2+1}$

4. $\lim_{x\to 1} \frac{x^3-1}{4x^3-x-3}$

5. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

6. $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1}$

Applying l'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 7–26.

7. $\lim_{t\to 0} \frac{\sin t^2}{t}$

8. $\lim_{x\to\pi/2} \frac{2x-\pi}{\cos x}$

9. $\lim_{\theta\to\pi} \frac{\sin\theta}{\pi-\theta}$

10. $\lim_{x\to\pi/2} \frac{1-\sin x}{1+\cos 2x}$

11. $\lim_{x\to\pi/4} \frac{\sin x-\cos x}{x-\pi/4}$

12. $\lim_{x\to\pi/3} \frac{\cos x-0.5}{x-\pi/3}$

13. $\lim_{x\to(\pi/2)^-} \left(x-\frac{\pi}{2}\right)\tan x$

14. $\lim_{x\to 0} \frac{2x}{x+7\sqrt{x}}$

15. $\lim_{x\to 1} \frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$

16. $\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$

17. $\lim_{x\to 0} \frac{\sqrt{a(a+x)}-a}{x}, \quad a>0$

18. $\lim_{t\to 0} \frac{10(\sin t-t)}{t^3}$

19. $\lim_{x\to 0} \frac{x(\cos x-1)}{\sin x-x}$

20. $\lim_{h\to 0} \frac{\sin(a+h)-\sin a}{h}$

21. $\lim_{r\to 1} \frac{a(r^n-1)}{r-1}$, n a positive integer

22. $\lim_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)$

23. $\lim_{x\to\infty} \left(x-\sqrt{x^2+x}\right)$

24. $\lim_{x\to\infty} x\tan\frac{1}{x}$

25. $\lim_{x\to\pm\infty} \frac{3x-5}{2x^2-x+2}$

26. $\lim_{x\to 0} \frac{\sin 7x}{\tan 11x}$

Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 27–30. Try it; you just keep on cycling. Find the limits some other way.

27. $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$

28. $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$

29. $\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x}$

30. $\lim_{x\to 0^+} \frac{\cot x}{\csc x}$

31. Which one is correct, and which one is wrong? Give reasons for your answers.

a. $\lim_{x\to 3} \frac{x-3}{x^2-3} = \lim_{x\to 3} \frac{1}{2x} = \frac{1}{6}$

b. $\lim_{x\to 3} \frac{x-3}{x^2-3} = \frac{0}{6} = 0$

32. ∞/∞ Form Give an example of two differentiable functions f and g with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ that satisfy the following.

a. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 3$

b. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

c. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

33. Continuous extension Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x-3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x=0$. Explain why your value of c works.

34. Let

$$f(x) = \begin{cases} x+2, & x\neq 0 \\ 0, & x=0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x+1, & x\neq 0 \\ 0, & x=0. \end{cases}$$

a. Show that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1 \quad \text{but} \quad \lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$$

b. Explain why this does not contradict l'Hôpital's Rule.

T **35. 0/0 Form** Estimate the value of

$$\lim_{x\to 1} \frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

T **36. $\infty - \infty$ Form**

a. Estimate the value of

$$\lim_{x\to\infty} \left(x-\sqrt{x^2+x}\right)$$

by graphing $f(x) = x-\sqrt{x^2+x}$ over a suitably large interval of x-values.

b. Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $\left(x+\sqrt{x^2+x}\right)/\left(x+\sqrt{x^2+x}\right)$ and simplify the new numerator.

T **37.** Let

$$f(x) = \frac{1 - \cos x^6}{x^{12}}.$$

Explain why some graphs of f may give false information about $\lim_{x\to 0} f(x)$. (*Hint:* Try the window $[-1, 1]$ by $[-0.5, 1]$.)

38. Find all values of c, that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

a. $f(x) = x, \quad g(x) = x^2, \quad (a, b) = (-2, 0)$

b. $f(x) = x, \quad g(x) = x^2, \quad (a, b)$ arbitrary

c. $f(x) = x^3/3 - 4x, \quad g(x) = x^2, \quad (a, b) = (0, 3)$

39. In the accompanying figure, the circle has radius OA equal to 1, and AB is tangent to the circle at A. The arc AC has radian measure θ and the segment AB also has length θ. The line through B and C crosses the x-axis at $P(x, 0)$.

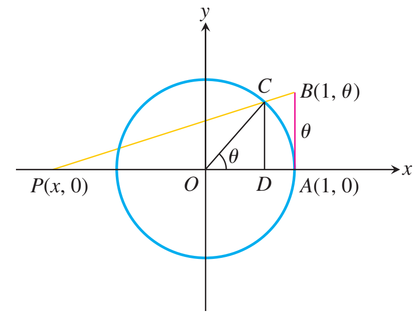
a. Show that the length of PA is

$$1 - x = \frac{\theta(1 - \cos\theta)}{\theta - \sin\theta}.$$

b. Find $\lim_{\theta\to 0} (1 - x)$.

c. Show that $\lim_{\theta\to\infty} [(1 - x) - (1 - \cos\theta)] = 0$.

Interpret this geometrically.



40. A right triangle has one leg of length 1, another of length y, and a hypotenuse of length r. The angle opposite y has radian measure θ. Find the limits as $\theta \to \pi/2$ of

a. $r - y$.

b. $r^2 - y^2$.

c. $r^3 - y^3$.