EXERCISES 4.7

Root-Finding

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find x_2.

2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.

3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2.

4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2.

5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.

6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.

Theory, Examples, and Applications

7. **Guessing a root** Suppose that your first guess is lucky, in the sense that x_0 is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to x_1 and later approximations?

8. **Estimating pi** You plan to estimate $\pi/2$ to five decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

9. **Oscillation** Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \ge 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

10. **Approximations that get worse and worse** Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate x_1, x_2, x_3, and x_4. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

11. Explain why the following four statements ask for the same information:

    i) Find the roots of $f(x) = x^3 - 3x - 1$.

    ii) Find the x-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.

    iii) Find the x-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.

    iv) Find the values of x where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

12. **Locating a planet** To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find x_1. (The value of the root is 1.49870 to five decimal places.) Remember to use radians.

T 13. **A program for using Newton's method on a grapher** Let $f(x) = x^3 + 3x + 1$. Here is a home screen program to perform the computations in Newton's method.

   a. Let $y_0 = f(x)$ and $y_1 = \text{NDER } f(x)$.

   b. Store $x_0 = -0.3$ into x.

   c. Then store $x - (y_0/y_1)$ into x and press the Enter key over and over. Watch as the numbers converge to the zero of f.

   d. Use different values for x_0 and repeat steps (b) and (c).

   e. Write your own equation and use this approach to solve it using Newton's method. Compare your answer with the answer given by the built-in feature of your calculator that gives zeros of functions.

T 14. (*Continuation of Exercise 11.*)

   a. Use Newton's method to find the two negative zeros of $f(x) = x^3 - 3x - 1$ to five decimal places.

   b. Graph $f(x) = x^3 - 3x - 1$ for $-2 \le x \le 2.5$. Use the Zoom and Trace features to estimate the zeros of f to five decimal places.

   c. Graph $g(x) = 0.25x^4 - 1.5x^2 - x + 5$. Use the Zoom and Trace features with appropriate rescaling to find, to five decimal places, the values of x where the graph has horizontal tangents.

T 15. **Intersecting curves** The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

T 16. **Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

T 17. a. How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?

   b. Use Newton's method to find them.

T 18. Intersection of curves

a. Does $\cos 3x$ ever equal x? Give reasons for your answer.

b. Use Newton's method to find where.

T 19. Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

T 20. Estimating pi Estimate π to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

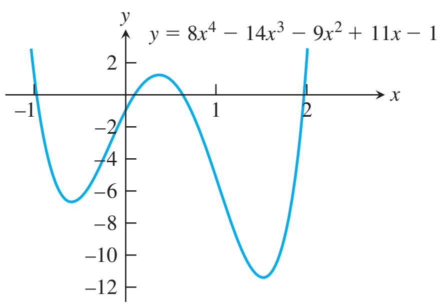
21. At what values(s) of x does $\cos x = 2x$?

22. At what value(s) of x does $\cos x = -x$?

23. Use the Intermediate Value Theorem from Section 2.6 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to five decimal places.

24. Factoring a quartic Find the approximate values of r_1 through r_4 in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$



T 25. Converging to different zeros Use Newton's method to find the zeros of $f(x) = 4x^4 - 4x^2$ using the given starting values (Figure 4.52).

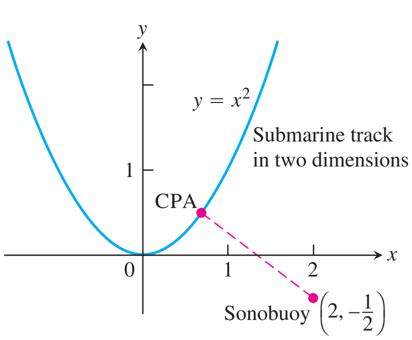
a. $x_0 = -2$ and $x_0 = -0.8$, lying in $\left(-\infty, -\sqrt{2}/2\right)$

b. $x_0 = -0.5$ and $x_0 = 0.25$, lying in $\left(-\sqrt{21/7}, \sqrt{21/7}\right)$

c. $x_0 = 0.8$ and $x_0 = 2$, lying in $\left(\sqrt{2}/2, \infty\right)$

d. $x_0 = -\sqrt{21/7}$ and $x_0 = \sqrt{21/7}$

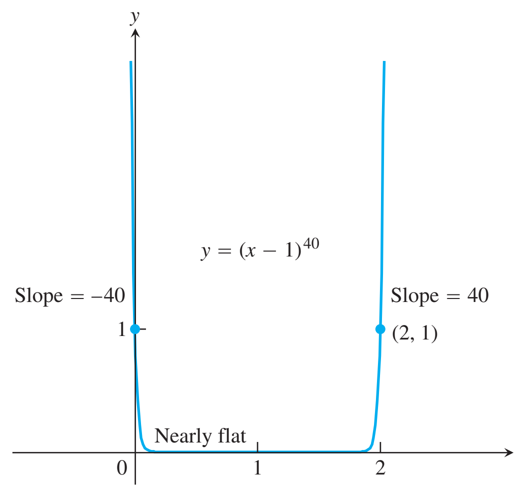
26. The sonobuoy problem In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.



a. Show that the value of x that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.

b. Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.

27. Curves that are nearly flat at the root Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$.



28. Finding a root different from the one sought All three roots of $f(x) = 4x^4 - 4x^2$ can be found by starting Newton's method near $x = \sqrt{21/7}$. Try it. (See Figure 4.52.)

29. Finding an ion concentration While trying to find the acidity of a saturated solution of magnesium hydroxide in hydrochloric acid, you derive the equation

$$\frac{3.64 \times 10^{-11}}{[H_3O^+]^2} = [H_3O^+] + 3.6 \times 10^{-4}$$

for the hydronium ion concentration $[H_3O^+]$. To find the value of $[H_3O^+]$, you set $x = 10^4[H_3O^+]$ and convert the equation to

$$x^3 + 3.6x^2 - 36.4 = 0.$$

You then solve this by Newton's method. What do you get for x? (Make it good to two decimal places.) For $[H_3O^+]$?

T 30. Complex roots If you have a computer or a calculator that can be programmed to do complex-number arithmetic, experiment with Newton's method to solve the equation $z^6 - 1 = 0$. The recursion relation to use is

$$z_{n+1} = z_n - \frac{z_n{}^6 - 1}{6z_n{}^5} \quad \text{or} \quad z_{n+1} = \frac{5}{6}z_n + \frac{1}{6z_n{}^5}.$$

Try these starting values (among others): $2, i, \sqrt{3} + i$.