4.8 Antiderivatives

We have studied how to find the derivative of a function. However, many problems require that we recover a function from its known derivative (from its known rate of change). For instance, we may know the velocity function of an object falling from an initial height and need to know its height at any time over some period. More generally, we want to find a function F from its derivative f. If such a function F exists, it is called an *antiderivative* of f.

Finding Antiderivatives

DEFINITION Antiderivative
A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE 1 Finding Antiderivatives

Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$

(b) $g(x) = \cos x$

(c) $h(x) = 2x + \cos x$

Solution

(a) $F(x) = x^2$

(b) $G(x) = \sin x$

(c) $H(x) = x^2 + \sin x$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$ and the derivative of $H(x) = x^2 + \sin x$ is $2x + \cos x$. ■

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C. Are there others?

Corollary 2 of the Mean Value Theorem in Section 4.2 gives the answer: Any two antiderivatives of a function differ by a constant. So the functions $x^2 + C$, where C is an **arbitrary constant**, form *all* the antiderivatives of $f(x) = 2x$. More generally, we have the following result.

> If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is
> $$F(x) + C$$
> where C is an arbitrary constant.

Thus the most general antiderivative of f on I is a *family* of functions $F(x) + C$ whose graphs are vertical translates of one another. We can select a particular antiderivative from this family by assigning a specific value to C. Here is an example showing how such an assignment might be made.

EXAMPLE 2 Finding a Particular Antiderivative

Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$.

Solution Since the derivative of $-\cos x$ is $\sin x$, the general antiderivative
$$F(x) = -\cos x + C$$
gives all the antiderivatives of $f(x)$. The condition $F(0) = 3$ determines a specific value for C. Substituting $x = 0$ into $F(x) = -\cos x + C$ gives
$$F(0) = -\cos 0 + C = -1 + C.$$
Since $F(0) = 3$, solving for C gives $C = 4$. So
$$F(x) = -\cos x + 4$$
is the antiderivative satisfying $F(0) = 3$. ■

By working backward from assorted differentiation rules, we can derive formulas and rules for antiderivatives. In each case there is an arbitrary constant C in the general expression representing all antiderivatives of a given function. Table 4.2 gives antiderivative formulas for a number of important functions.

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$, n rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

The rules in Table 4.2 are easily verified by differentiating the general antiderivative formula to obtain the function to its left. For example, the derivative of $\tan x + C$ is $\sec^2 x$, whatever the value of the constant C, and this establishes the formula for the most general antiderivative of $\sec^2 x$.

EXAMPLE 3 Finding Antiderivatives Using Table 4.2

Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$

(b) $g(x) = \dfrac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \dfrac{x}{2}$

Solution

(a) $F(x) = \dfrac{x^6}{6} + C$ — Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$ — Formula 1 with $n = -1/2$

(c) $H(x) = \dfrac{-\cos 2x}{2} + C$ — Formula 2 with $k = 2$

(d) $I(x) = \dfrac{\sin (x/2)}{1/2} + C = 2 \sin \dfrac{x}{2} + C$ — Formula 3 with $k = 1/2$ ■

Other derivative rules also lead to corresponding antiderivative rules. We can add and subtract antiderivatives, and multiply them by constants.

The formulas in Table 4.3 are easily proved by differentiating the antiderivatives and verifying that the result agrees with the original function. Formula 2 is the special case $k = -1$ in Formula 1.

TABLE 4.3 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, k a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$,
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

EXAMPLE 4 Using the Linearity Rules for Antiderivatives

Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$

Solution We have that $f(x) = 3g(x) + h(x)$ for the functions g and h in Example 3. Since $G(x) = 2\sqrt{x}$ is an antiderivative of $g(x)$ from Example 3b, it follows from the Constant Multiple Rule for antiderivatives that $3G(x) = 3 \cdot 2\sqrt{x} = 6\sqrt{x}$ is an antiderivative of $3g(x) = 3/\sqrt{x}$. Likewise, from Example 3c we know that $H(x) = (-1/2)\cos 2x$ is an antiderivative of $h(x) = \sin 2x$. From the Sum Rule for antiderivatives, we then get that

$$\begin{aligned} F(x) &= 3G(x) + H(x) + C \\ &= 6\sqrt{x} - \frac{1}{2}\cos 2x + C \end{aligned}$$

is the general antiderivative formula for $f(x)$, where C is an arbitrary constant. ■

Antiderivatives play several important roles, and methods and techniques for finding them are a major part of calculus. (This is the subject of Chapter 8.)

Initial Value Problems and Differential Equations

Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

This is called a **differential equation**, since it is an equation involving an unknown function y that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition

$$y(x_0) = y_0.$$

This condition means the function $y(x)$ has the value y_0 when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**. Such problems play important roles in all branches of science. Here's an example of solving an initial value problem.

EXAMPLE 5 Finding a Curve from Its Slope Function and a Point

Find the curve whose slope at the point (x, y) is $3x^2$ if the curve is required to pass through the point $(1, -1)$.

Solution In mathematical language, we are asked to solve the initial value problem that consists of the following.

The differential equation: $\dfrac{dy}{dx} = 3x^2$ The curve's slope is $3x^2$.

The initial condition: $y(1) = -1$

1. *Solve the differential equation*: The function y is an antiderivative of $f(x) = 3x^2$, so

$$y = x^3 + C.$$

This result tells us that y equals $x^3 + C$ for some value of C. We find that value from the initial condition $y(1) = -1$.

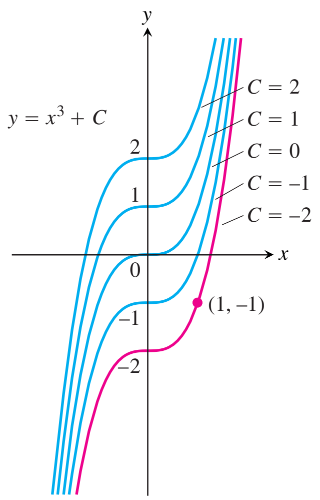

FIGURE 4.54 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 5, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

Video

You Try It

2. *Evaluate C*:

$$\begin{aligned} y &= x^3 + C \\ -1 &= (1)^3 + C \qquad \text{Initial condition } y(1) = -1 \\ C &= -2. \end{aligned}$$

The curve we want is $y = x^3 - 2$ (Figure 4.54). ■

The most general antiderivative $F(x) + C$ (which is $x^3 + C$ in Example 5) of the function $f(x)$ gives the **general solution** $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives *all* the solutions of the equation (there are infinitely many, one for each value of C). We **solve** the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$.

Antiderivatives and Motion

We have seen that the derivative of the position of an object gives its velocity, and the derivative of its velocity gives its acceleration. If we know an object's acceleration, then by finding an antiderivative we can recover the velocity, and from an antiderivative of the velocity we can recover its position function. This procedure was used as an application of Corollary 2 in Section 4.2. Now that we have a terminology and conceptual framework in terms of antiderivatives, we revisit the problem from the point of view of differential equations.

EXAMPLE 6 Dropping a Package from an Ascending Balloon

A balloon ascending at the rate of 12 ft/sec is at a height 80 ft above the ground when a package is dropped. How long does it take the package to reach the ground?

Solution Let $v(t)$ denote the velocity of the package at time t, and let $s(t)$ denote its height above the ground. The acceleration of gravity near the surface of the earth is 32 ft/sec^2. Assuming no other forces act on the dropped package, we have

$$\frac{dv}{dt} = -32. \qquad \text{Negative because gravity acts in the direction of decreasing } s.$$

This leads to the initial value problem.

$$\begin{aligned} &\textit{Differential equation:} \quad \frac{dv}{dt} = -32 \\ &\textit{Initial condition:} \quad v(0) = 12, \end{aligned}$$

which is our mathematical model for the package's motion. We solve the initial value problem to obtain the velocity of the package.

1. *Solve the differential equation*: The general formula for an antiderivative of -32 is

$$v = -32t + C.$$

Having found the general solution of the differential equation, we use the initial condition to find the particular solution that solves our problem.

2. *Evaluate C*:

$$\begin{aligned} 12 &= -32(0) + C \qquad \text{Initial condition } v(0) = 12 \\ C &= 12. \end{aligned}$$

The solution of the initial value problem is

$$v = -32t + 12.$$

Since velocity is the derivative of height and the height of the package is 80 ft at the time $t = 0$ when it is dropped, we now have a second initial value problem.

$$\textit{Differential equation:} \quad \frac{ds}{dt} = -32t + 12 \qquad \text{Set } v = ds/dt \text{ in the last equation.}$$

$$\textit{Initial condition:} \quad s(0) = 80$$

We solve this initial value problem to find the height as a function of t.

1. *Solve the differential equation*: Finding the general antiderivative of $-32t + 12$ gives

$$s = -16t^2 + 12t + C.$$

2. *Evaluate C*:

$$\begin{aligned} 80 &= -16(0)^2 + 12(0) + C \qquad \text{Initial condition } s(0) = 80 \\ C &= 80. \end{aligned}$$

The package's height above ground at time t is

$$s = -16t^2 + 12t + 80.$$

Use the solution: To find how long it takes the package to reach the ground, we set s equal to 0 and solve for t:

$$\begin{aligned} -16t^2 + 12t + 80 &= 0 \\ -4t^2 + 3t + 20 &= 0 \\ t &= \frac{-3 \pm \sqrt{329}}{-8} \qquad \text{Quadratic formula} \\ t \approx -1.89, &\qquad t \approx 2.64. \end{aligned}$$

The package hits the ground about 2.64 sec after it is dropped from the balloon. (The negative root has no physical meaning.) ■

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f.

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2x\,dx = x^2 + C,$$

$$\int \cos x\,dx = \sin x + C,$$

$$\int (2x + \cos x)\,dx = x^2 + \sin x + C.$$

This notation is related to the main application of antiderivatives, which will be explored in Chapter 5. Antiderivatives play a key role in computing limits of infinite sums, an unexpected and wonderfully useful role that is described in a central result of Chapter 5, called the Fundamental Theorem of Calculus.

You Try It

EXAMPLE 7 Indefinite Integration Done Term-by-Term and Rewriting the Constant of Integration

Evaluate

$$\int (x^2 - 2x + 5)\,dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\ &= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx \\ &= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the antiderivatives there are. For this reason, we recommend that you go right to the final form even if you elect to integrate term-by-term. Write

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$

Find the simplest antiderivative you can for each part and add the arbitrary constant of integration at the end. ■