Chapter 4 Practice Exercises

Existence of Extreme Values

1. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.

2. Does $g(x) = \csc x + 2 \cot x$ have any local maximum values? Give reasons for your answer.

3. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f.

4. Find values of a and b such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

5. The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of x, assumes a local maximum value of 0 at each point of $[0, 1)$. Could any of these local maximum values also be local minimum values of f? Give reasons for your answer.

6. **a.** Give an example of a differentiable function f whose first derivative is zero at some point c even though f has neither a local maximum nor a local minimum at c.

b. How is this consistent with Theorem 2 in Section 4.1? Give reasons for your answer.

7. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Extreme Value Theorem for continuous functions? Why?

8. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Notice that the interval is not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

T 9. A graph that is large enough to show a function's global behavior may fail to reveal important local features. The graph of $f(x) = (x^8/8) - (x^6/2) - x^5 + 5x^3$ is a case in point.

a. Graph f over the interval $-2.5 \le x \le 2.5$. Where does the graph appear to have local extreme values or points of inflection?

b. Now factor $f'(x)$ and show that f has a local maximum at $x = \sqrt[3]{5} \approx 1.70998$ and local minima at $x = \pm\sqrt{3} \approx \pm 1.73205$.

c. Zoom in on the graph to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt{3}$.

The moral here is that without calculus the existence of two of the three extreme values would probably have gone unnoticed. On any normal graph of the function, the values would lie close enough together to fall within the dimensions of a single pixel on the screen.

(*Source*: *Uses of Technology in the Mathematics Curriculum*, by Benny Evans and Jerry Johnson, Oklahoma State University, published in 1990 under National Science Foundation Grant USE-8950044.)

T 10. (*Continuation of Exercise 9.*)

a. Graph $f(x) = (x^8/8) - (2/5)x^5 - 5x - (5/x^2) + 11$ over the interval $-2 \le x \le 2$. Where does the graph appear to have local extreme values or points of inflection?

b. Show that f has a local maximum value at $x = \sqrt[7]{5} \approx 1.2585$ and a local minimum value at $x = \sqrt[3]{2} \approx 1.2599$.

c. Zoom in to find a viewing window that shows the presence of the extreme values at $x = \sqrt[7]{5}$ and $x = \sqrt[3]{2}$.

The Mean Value Theorem

11. **a.** Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.

b. How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.

12. **a.** Show that $y = \tan\theta$ increases on every interval in its domain.

b. If the conclusion in part (a) is really correct, how do you explain the fact that $\tan\pi = 0$ is less than $\tan(\pi/4) = 1$?

13. **a.** Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on $[0, 1]$.

T **b.** Find the solution to as many decimal places as you can.

14. **a.** Show that $f(x) = x/(x + 1)$ increases on every interval in its domain.

b. Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.

15. **Water in a reservoir** As a result of a heavy rain, the volume of water in a reservoir increased by 1400 acre-ft in 24 hours. Show that at some instant during that period the reservoir's volume was increasing at a rate in excess of 225,000 gal/min. (An acre-foot is 43,560 ft^3, the volume that would cover 1 acre to the depth of 1 ft. A cubic foot holds 7.48 gal.)

16. The formula $F(x) = 3x + C$ gives a different function for each value of C. All of these functions, however, have the same derivative with respect to x, namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

17. Show that

$$\frac{d}{dx}\left(\frac{x}{x + 1}\right) = \frac{d}{dx}\left(-\frac{1}{x + 1}\right)$$

even though

$$\frac{x}{x + 1} \ne -\frac{1}{x + 1}.$$

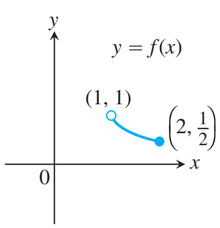
Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

18. Calculate the first derivatives of $f(x) = x^2/(x^2 + 1)$ and $g(x) = -1/(x^2 + 1)$. What can you conclude about the graphs of these functions?

Conclusions from Graphs

In Exercises 19 and 20, use the graph to answer the questions.

19. Identify any global extreme values of f and the values of x at which they occur.

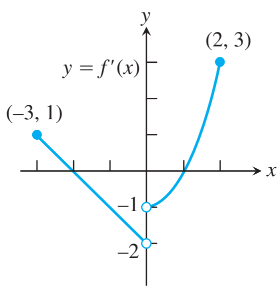


20. Estimate the intervals on which the function $y = f(x)$ is

a. increasing.

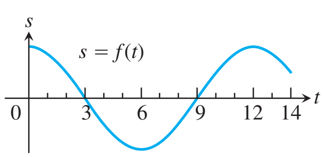
b. decreasing.

c. Use the given graph of f' to indicate where any local extreme values of the function occur, and whether each extreme is a relative maximum or minimum.

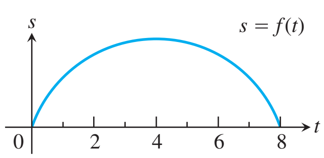


Each of the graphs in Exercises 21 and 22 is the graph of the position function $s = f(t)$ of a body moving on a coordinate line (t represents time). At approximately what times (if any) is each body's **(a)** velocity equal to zero? **(b)** Acceleration equal to zero? During approximately what time intervals does the body move **(c)** forward? **(d)** Backward?

21.



22.



Graphs and Graphing

Graph the curves in Exercises 23–32.

23. $y = x^2 - (x^3/6)$ **24.** $y = x^3 - 3x^2 + 3$

25. $y = -x^3 + 6x^2 - 9x + 3$

26. $y = (1/8)(x^3 + 3x^2 - 9x - 27)$

27. $y = x^3(8 - x)$ **28.** $y = x^2(2x^2 - 9)$

29. $y = x - 3x^{2/3}$ **30.** $y = x^{1/3}(x - 4)$

31. $y = x\sqrt{3 - x}$ **32.** $y = x\sqrt{4 - x^2}$

Each of Exercises 33–38 gives the first derivative of a function $y = f(x)$. **(a)** At what points, if any, does the graph of f have a local maximum, local minimum, or inflection point? **(b)** Sketch the general shape of the graph.

33. $y' = 16 - x^2$ **34.** $y' = x^2 - x - 6$

35. $y' = 6x(x + 1)(x - 2)$ **36.** $y' = x^2(6 - 4x)$

37. $y' = x^4 - 2x^2$ **38.** $y' = 4x^2 - x^4$

In Exercises 39–42, graph each function. Then use the function's first derivative to explain what you see.

39. $y = x^{2/3} + (x - 1)^{1/3}$ **40.** $y = x^{2/3} + (x - 1)^{2/3}$

41. $y = x^{1/3} + (x - 1)^{1/3}$ **42.** $y = x^{2/3} - (x - 1)^{1/3}$

Sketch the graphs of the functions in Exercises 43–50.

43. $y = \dfrac{x + 1}{x - 3}$ **44.** $y = \dfrac{2x}{x + 5}$

45. $y = \dfrac{x^2 + 1}{x}$ **46.** $y = \dfrac{x^2 - x + 1}{x}$

47. $y = \dfrac{x^3 + 2}{2x}$ **48.** $y = \dfrac{x^4 - 1}{x^2}$

49. $y = \dfrac{x^2 - 4}{x^2 - 3}$ **50.** $y = \dfrac{x^2}{x^2 - 4}$

Applying l'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 51–62.

51. $\lim_{x \to 1} \dfrac{x^2 + 3x - 4}{x - 1}$ **52.** $\lim_{x \to 1} \dfrac{x^a - 1}{x^b - 1}$

53. $\lim_{x \to \pi} \dfrac{\tan x}{x}$ **54.** $\lim_{x \to 0} \dfrac{\tan x}{x + \sin x}$

55. $\lim_{x \to 0} \dfrac{\sin^2 x}{\tan(x^2)}$ **56.** $\lim_{x \to 0} \dfrac{\sin mx}{\sin nx}$

57. $\lim_{x \to \pi/2^-} \sec 7x \cos 3x$ **58.** $\lim_{x \to 0^+} \sqrt{x} \sec x$

59. $\lim_{x \to 0} (\csc x - \cot x)$ **60.** $\lim_{x \to 0} \left(\dfrac{1}{x^4} - \dfrac{1}{x^2}\right)$

61. $\lim_{x \to \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right)$

62. $\lim_{x \to \infty} \left(\dfrac{x^3}{x^2 - 1} - \dfrac{x^3}{x^2 + 1}\right)$

Optimization

63. The sum of two nonnegative numbers is 36. Find the numbers if

a. the difference of their square roots is to be as large as possible.

b. the sum of their square roots is to be as large as possible.

64. The sum of two nonnegative numbers is 20. Find the numbers

a. if the product of one number and the square root of the other is to be as large as possible.

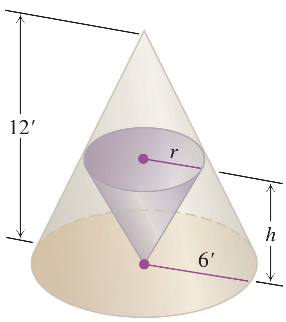
b. if one number plus the square root of the other is to be as large as possible.

65. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

66. A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 ft^3, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

67. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

68. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



69. Manufacturing tires Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

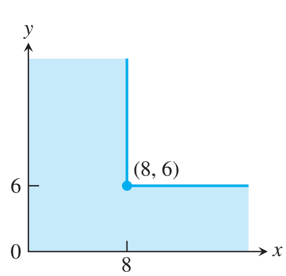
70. Particle motion The positions of two particles on the s-axis are $s_1 = \cos t$ and $s_2 = \cos(t + \pi/4)$.

a. What is the farthest apart the particles ever get?

b. When do the particles collide?

T **71. Open-top box** An open-top rectangular box is constructed from a 10-in.-by-16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

72. The ladder problem What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.



Newton's Method

73. Let $f(x) = 3x - x^3$. Show that the equation $f(x) = -4$ has a solution in the interval [2, 3] and use Newton's method to find it.

74. Let $f(x) = x^4 - x^3$. Show that the equation $f(x) = 75$ has a solution in the interval [3, 4] and use Newton's method to find it.

Finding Indefinite Integrals

Find the indefinite integrals (most general antiderivatives) in Exercises 75–90. Check your answers by differentiation.

75. $\int (x^3 + 5x - 7)\,dx$

76. $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

77. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

78. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

79. $\int \frac{dr}{(r + 5)^2}$

80. $\int \frac{6\,dr}{\left(r - \sqrt{2}\right)^3}$

81. $\int 3\theta\sqrt{\theta^2 + 1}\,d\theta$

82. $\int \frac{\theta}{\sqrt{7 + \theta^2}}\,d\theta$

83. $\int x^3(1 + x^4)^{-1/4}\,dx$

84. $\int (2 - x)^{3/5}\,dx$

85. $\int \sec^2 \frac{s}{10}\,ds$

86. $\int \csc^2 \pi s\,ds$

87. $\int \csc\sqrt{2}\theta \cot\sqrt{2}\theta\,d\theta$

88. $\int \sec\frac{\theta}{3}\tan\frac{\theta}{3}\,d\theta$

89. $\int \sin^2 \frac{x}{4}\,dx$

90. $\int \cos^2 \frac{x}{2}\,dx \left(\textit{Hint: } \cos^2\theta = \frac{1 + \cos 2\theta}{2}\right)$

Initial Value Problems

Solve the initial value problems in Exercises 91–94.

91. $\frac{dy}{dx} = \frac{x^2 + 1}{x^2}, \quad y(1) = -1$

92. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2, \quad y(1) = 1$

93. $\frac{d^2r}{dt^2} = 15\sqrt{t} + \frac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

94. $\frac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$