EXERCISES 5.3

Expressing Limits as Integrals

Express the limits in Exercises 1–8 as definite integrals.

1. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} c_k^2 \, \Delta x_k$, where P is a partition of $[0, 2]$

2. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} 2c_k^3 \, \Delta x_k$, where P is a partition of $[-1, 0]$

3. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k) \, \Delta x_k$, where P is a partition of $[-7, 5]$

4. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k$, where P is a partition of $[1, 4]$

5. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k} \, \Delta x_k$, where P is a partition of $[2, 3]$

6. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2} \, \Delta x_k$, where P is a partition of $[0, 1]$

7. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\sec c_k) \, \Delta x_k$, where P is a partition of $[-\pi/4, 0]$

8. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\tan c_k) \, \Delta x_k$, where P is a partition of $[0, \pi/4]$

Using Properties and Known Values to Find Other Integrals

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4, \int_1^5 f(x)\,dx = 6, \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.3 to find

a. $\int_2^2 g(x)\,dx$ **b.** $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ **d.** $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ **f.** $\int_1^5 [4f(x) - g(x)]\,dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.3 to find

a. $\int_1^9 -2f(x)\,dx$ **b.** $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ **d.** $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ **f.** $\int_9^7 [h(x) - f(x)]\,dx$

11. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ **b.** $\int_1^2 \sqrt{3}f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ **d.** $\int_1^2 [-f(x)]\,dx$

12. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a. $\int_0^{-3} g(t)\,dt$ **b.** $\int_{-3}^0 g(u)\,du$

c. $\int_{-3}^0 [-g(x)]\,dx$ **d.** $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

13. Suppose that f is integrable and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a. $\int_3^4 f(z)\,dz$ **b.** $\int_4^3 f(t)\,dt$

14. Suppose that h is integrable and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a. $\int_1^3 h(r)\,dr$ **b.** $-\int_3^1 h(u)\,du$

Using Area to Evaluate Definite Integrals

In Exercises 15–22, graph the integrands and use areas to evaluate the integrals.

15. $\int_{-2}^4 \left(\frac{x}{2} + 3\right)dx$ **16.** $\int_{1/2}^{3/2} (-2x + 4)\,dx$

17. $\int_{-3}^3 \sqrt{9 - x^2}\,dx$ **18.** $\int_{-4}^0 \sqrt{16 - x^2}\,dx$

19. $\int_{-2}^1 |x|\,dx$ **20.** $\int_{-1}^1 (1 - |x|)\,dx$

21. $\int_{-1}^1 (2 - |x|)\,dx$ **22.** $\int_{-1}^1 \left(1 + \sqrt{1 - x^2}\right)dx$

Use areas to evaluate the integrals in Exercises 23–26.

23. $\int_0^b \frac{x}{2}\,dx, \quad b > 0$ **24.** $\int_0^b 4x\,dx, \quad b > 0$

25. $\int_a^b 2s\,ds, \quad 0 < a < b$ **26.** $\int_a^b 3t\,dt, \quad 0 < a < b$

Evaluations

Use the results of Equations (1) and (3) to evaluate the integrals in Exercises 27–38.

27. $\int_1^{\sqrt{2}} x\,dx$ **28.** $\int_{0.5}^{2.5} x\,dx$ **29.** $\int_\pi^{2\pi} \theta\,d\theta$

30. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$ **31.** $\int_0^{\sqrt[3]{7}} x^2\,dx$ **32.** $\int_0^{0.3} s^2\,ds$

33. $\int_0^{1/2} t^2\,dt$ **34.** $\int_0^{\pi/2} \theta^2\,d\theta$ **35.** $\int_a^{2a} x\,dx$

36. $\int_a^{\sqrt{3}a} x\,dx$ **37.** $\int_0^{\sqrt[3]{b}} x^2\,dx$ **38.** $\int_0^{3b} x^2\,dx$

Use the rules in Table 5.3 and Equations (1)–(3) to evaluate the integrals in Exercises 39–50.

39. $\int_3^1 7\,dx$ **40.** $\int_0^{-2} \sqrt{2}\,dx$

41. $\int_0^2 5x\,dx$ **42.** $\int_3^5 \frac{x}{8}\,dx$

43. $\int_0^2 (2t - 3)\,dt$ **44.** $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right)dt$

45. $\int_2^1 \left(1 + \frac{z}{2}\right)dz$ **46.** $\int_3^0 (2z - 3)\,dz$

47. $\int_1^2 3u^2\,du$ **48.** $\int_{1/2}^1 24u^2\,du$

49. $\int_0^2 (3x^2 + x - 5)\,dx$ **50.** $\int_1^0 (3x^2 + x - 5)\,dx$

Finding Area

In Exercises 51–54 use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$.

51. $y = 3x^2$ **52.** $y = \pi x^2$

53. $y = 2x$ **54.** $y = \frac{x}{2} + 1$

Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1$ on $\left[0, \sqrt{3}\right]$

56. $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$ **57.** $f(x) = -3x^2 - 1$ on $[0, 1]$

58. $f(x) = 3x^2 - 3$ on $[0, 1]$

59. $f(t) = (t - 1)^2$ on $[0, 3]$

60. $f(t) = t^2 - t$ on $[-2, 1]$

61. $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

62. $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

Theory and Examples

63. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\, dx?$$

(*Hint:* Where is the integrand positive?)

64. What values of a and b minimize the value of

$$\int_a^b (x^4 - 2x^2)\, dx?$$

65. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}\, dx.$$

66. (*Continuation of Exercise 65*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2}\, dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1 + x^2}\, dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2}\, dx.$$

67. Show that the value of $\int_0^1 \sin(x^2)\, dx$ cannot possibly be 2.

68. Show that the value of $\int_1^0 \sqrt{x + 8}\, dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

69. Integrals of nonnegative functions Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \ge 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\, dx \ge 0.$$

70. Integrals of nonpositive functions Show that if f is integrable then

$$f(x) \le 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\, dx \le 0.$$

71. Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\, dx$.

72. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\, dx$.

73. If $\text{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $\text{av}(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b \text{av}(f)\, dx = \int_a^b f(x)\, dx?$$

Give reasons for your answer.

74. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

a. $\text{av}(f + g) = \text{av}(f) + \text{av}(g)$

b. $\text{av}(kf) = k\,\text{av}(f)$ (any number k)

c. $\text{av}(f) \le \text{av}(g)$ if $f(x) \le g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

75. Use limits of Riemann sums as in Example 4a to establish Equation (2).

76. Use limits of Riemann sums as in Example 4a to establish Equation (3).

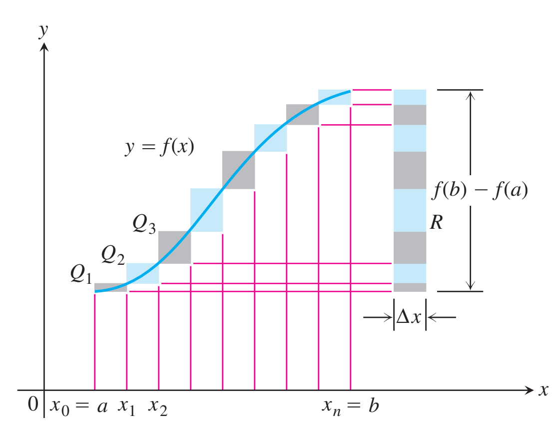
77. Upper and lower sums for increasing functions

a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint*: The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \le |f(b) - f(a)|\, \Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P, and hence that $\lim_{\|P\| \to 0} (U - L) = 0$.



78. Upper and lower sums for decreasing functions (*Continuation of Exercise 77*)

a. Draw a figure like the one in Exercise 77 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 77a.

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \leq |f(b) - f(a)|\ \Delta x_{\max}$$

of Exercise 77b still holds and hence that $\lim_{\|P\| \to 0} (U - L) = 0$.

79. Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2 \sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

a. Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b. Find the limit of U as $n \to \infty$ and $\Delta x = (b - a)/n \to 0$.

80. Suppose that f is continuous and nonnegative over $[a, b]$, as in the figure at the right. By inserting points

$$x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a$, $\Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = b - x_{n-1}$, which need not be equal.

a. If $m_k = \min\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the *lower sum*

$$L = m_1\,\Delta x_1 + m_2\,\Delta x_2 + \cdots + m_n\,\Delta x_n$$

and the shaded region in the first part of the figure.

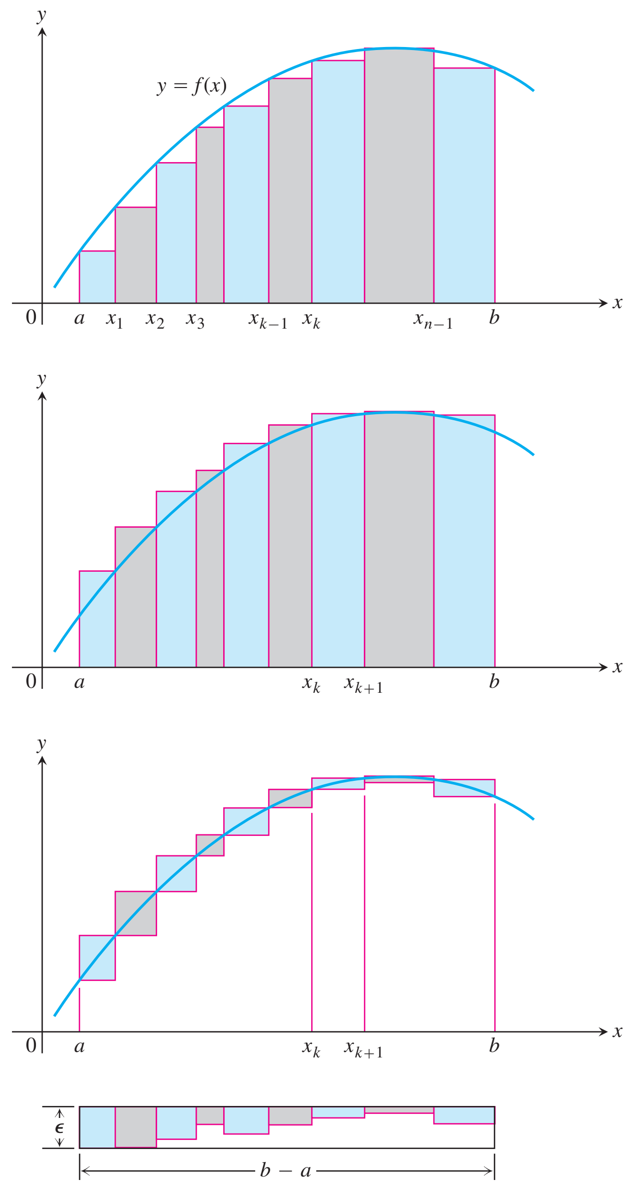
b. If $M_k = \max\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the *upper sum*

$$U = M_1\,\Delta x_1 + M_2\,\Delta x_2 + \cdots + M_n\,\Delta x_n$$

and the shaded region in the second part of the figure.

c. Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.

81. We say f is **uniformly continuous** on $[a, b]$ if given any $\epsilon > 0$ there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| < \epsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure at the right to show that if f is continuous and $\epsilon > 0$ is given, it is possible to make $U - L \leq \epsilon \cdot (b - a)$ by making the largest of the Δx_k's sufficiently small.



82. If you average 30 mi/h on a 150-mi trip and then return over the same 150 mi at the rate of 50 mi/h, what is your average speed for the trip? Give reasons for your answer. (Source: David H. Pleacher, *The Mathematics Teacher*, Vol. 85, No. 6, pp. 445–446, September 1992.)

COMPUTER EXPLORATIONS

Finding Riemann Sums

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 83–88. Use $n = 4, 10, 20$, and 50 subintervals of equal length in each case.

83. $\displaystyle\int_0^1 (1 - x)\,dx = \frac{1}{2}$

84. $\displaystyle\int_0^1 (x^2 + 1)\,dx = \frac{4}{3}$

85. $\int_{-\pi}^{\pi} \cos x \, dx = 0$

86. $\int_{0}^{\pi/4} \sec^2 x \, dx = 1$

87. $\int_{-1}^{1} |x| \, dx = 1$

88. $\int_{1}^{2} \frac{1}{x} \, dx$ (The integral's value is about 0.693.)

Average Value

In Exercises 89–92, use a CAS to perform the following steps:

a. Plot the functions over the given interval.

b. Partition the interval into $n = 100, 200$, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

c. Compute the average value of the function values generated in part (b).

d. Solve the equation $f(x) =$ (average value) for x using the average value calculated in part (c) for the $n = 1000$ partitioning.

89. $f(x) = \sin x \quad \text{on} \quad [0, \pi]$

90. $f(x) = \sin^2 x \quad \text{on} \quad [0, \pi]$

91. $f(x) = x \sin \frac{1}{x} \quad \text{on} \quad \left[\frac{\pi}{4}, \pi\right]$

92. $f(x) = x \sin^2 \frac{1}{x} \quad \text{on} \quad \left[\frac{\pi}{4}, \pi\right]$