5.4 The Fundamental Theorem of Calculus

HISTORICAL BIOGRAPHY

Sir Isaac Newton
(1642–1727)

In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than by taking limits of Riemann sums as we did in Section 5.3. Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years.

Along the way, we present the integral version of the Mean Value Theorem, which is another important theorem of integral calculus and used to prove the Fundamental Theorem.

Mean Value Theorem for Definite Integrals

In the previous section, we defined the average value of a continuous function over a closed interval $[a, b]$ as the definite integral $\int_a^b f(x)\,dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.16 shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.

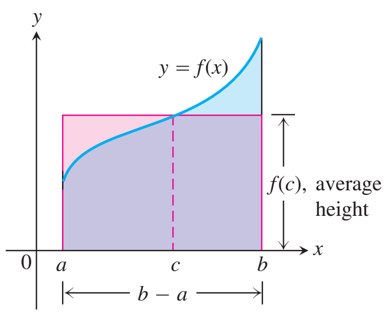

FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\,dx.$$

THEOREM 3 The Mean Value Theorem for Definite Integrals
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Proof If we divide both sides of the Max-Min Inequality (Table 5.3, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 2.6) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $(1/(b-a)) \int_a^b f(x)\,dx$ at some point c in $[a, b]$. ■

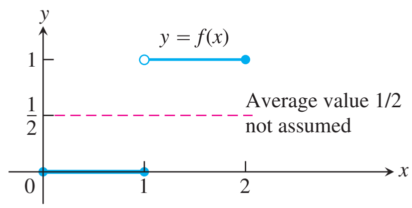

FIGURE 5.17 A discontinuous function need not assume its average value.

The continuity of f is important here. It is possible that a discontinuous function never equals its average value (Figure 5.17).

EXAMPLE 1 Applying the Mean Value Theorem for Integrals

Find the average value of $f(x) = 4 - x$ on $[0, 3]$ and where f actually takes on this value at some point in the given domain.

Solution

$$\begin{aligned} \text{av}(f) &= \frac{1}{b-a} \int_a^b f(x)\,dx \\ &= \frac{1}{3-0} \int_0^3 (4-x)\,dx = \frac{1}{3}\left(\int_0^3 4\,dx - \int_0^3 x\,dx\right) \\ &= \frac{1}{3}\left(4(3-0) - \left(\frac{3^2}{2} - \frac{0^2}{2}\right)\right) \qquad \text{Section 5.3, Eqs. (1) and (2)} \\ &= 4 - \frac{3}{2} = \frac{5}{2}. \end{aligned}$$

The average value of $f(x) = 4 - x$ over $[0, 3]$ is $5/2$. The function assumes this value when $4 - x = 5/2$ or $x = 3/2$. (Figure 5.18) ■

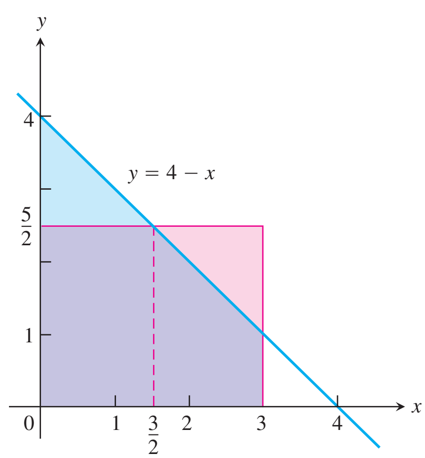

FIGURE 5.18 The area of the rectangle with base $[0, 3]$ and height $5/2$ (the average value of the function $f(x) = 4 - x$) is equal to the area between the graph of f and the x-axis from 0 to 3 (Example 1).

In Example 1, we actually found a point c where f assumed its average value by setting $f(x)$ equal to the calculated average value and solving for x. It's not always possible to solve easily for the value c. What else can we learn from the Mean Value Theorem for integrals? Here's an example.

EXAMPLE 2 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x)\,dx = \frac{1}{b-a} \cdot 0 = 0.$$

By the Mean Value Theorem, f assumes this value at some point $c \in [a, b]$. ■

Fundamental Theorem, Part 1

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

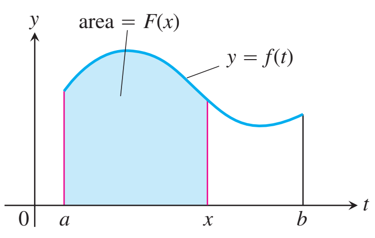

FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

For example, if f is nonnegative and x lies to the right of a, then $F(x)$ is the area under the graph from a to x (Figure 5.19). The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x, there is a well-defined numerical output, in this case the definite integral of f from a to x.

Equation (1) gives a way to define new functions, but its importance now is the connection it makes between integrals and derivatives. If f is any continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. At every value of x,

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

To gain some insight into why this result holds, we look at the geometry behind it.

If $f \ge 0$ on $[a, b]$, then the computation of $F'(x)$ from the definition of the derivative means taking the limit as $h \to 0$ of the difference quotient

$$\frac{F(x+h) - F(x)}{h}.$$

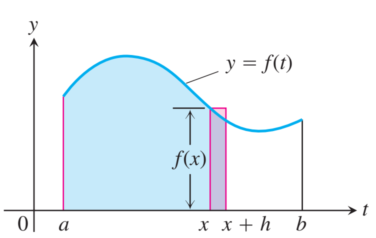

FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x+h)$ is the area to the left of $x + h$. The difference quotient $[F(x+h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

For $h > 0$, the numerator is obtained by subtracting two areas, so it is the area under the graph of f from x to $x + h$ (Figure 5.20). If h is small, this area is approximately equal to the area of the rectangle of height $f(x)$ and width h, which can be seen from Figure 5.20. That is,

$$F(x+h) - F(x) \approx hf(x).$$

Dividing both sides of this approximation by h and letting $h \to 0$, it is reasonable to expect that

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = f(x).$$

This result is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus.

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Before proving Theorem 4, we look at several examples to gain a better understanding of what it says.

EXAMPLE 3 Applying the Fundamental Theorem

Use the Fundamental Theorem to find

(a) $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\, dt$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^2}\, dt$

(c) $\dfrac{dy}{dx}$ if $y = \displaystyle\int_x^5 3t \sin t\, dt$

(d) $\dfrac{dy}{dx}$ if $y = \displaystyle\int_1^{x^2} \cos t\, dt$

Solution

(a) $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\, dt = \cos x$ Eq. 2 with $f(t) = \cos t$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^2}\, dt = \frac{1}{1+x^2}$ Eq. 2 with $f(t) = \dfrac{1}{1+t^2}$

(c) Rule 1 for integrals in Table 5.3 of Section 5.3 sets this up for the Fundamental Theorem.

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t\, dt = \frac{d}{dx}\left(-\int_5^x 3t\sin t\, dt\right) \qquad \text{Rule 1}$$

$$= -\frac{d}{dx}\int_5^x 3t\sin t\, dt$$

$$= -3x\sin x$$

(d) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t\, dt \qquad \text{and} \qquad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \left(\frac{d}{du}\int_1^u \cos t\, dt\right)\cdot\frac{du}{dx}$$

$$= \cos u \cdot \frac{du}{dx}$$

$$= \cos(x^2)\cdot 2x$$

$$= 2x\cos x^2$$

■

EXAMPLE 4 Constructing a Function with a Given Derivative and Value

Find a function $y = f(x)$ on the domain $(-\pi/2, \pi/2)$ with derivative

$$\frac{dy}{dx} = \tan x$$

that satisfies the condition $f(3) = 5$.

Solution The Fundamental Theorem makes it easy to construct a function with derivative $\tan x$ that equals 0 at $x = 3$:

$$y = \int_3^x \tan t\, dt.$$

Since $y(3) = \int_3^3 \tan t\, dt = 0$, we have only to add 5 to this function to construct one with derivative $\tan x$ whose value at $x = 3$ is 5:

$$f(x) = \int_3^x \tan t\, dt + 5.$$

Although the solution to the problem in Example 4 satisfies the two required conditions, you might ask whether it is in a useful form. The answer is yes, since today we have computers and calculators that are capable of approximating integrals. In Chapter 7 we will learn to write the solution in Example 4 exactly as

$$y = \ln\left|\frac{\cos 3}{\cos x}\right| + 5.$$

We now give a proof of the Fundamental Theorem for an arbitrary continuous function.

Proof of Theorem 4 We prove the Fundamental Theorem by applying the definition of the derivative directly to the function $F(x)$, when x and $x + h$ are in (a, b). This means writing out the difference quotient

$$\frac{F(x + h) - F(x)}{h} \tag{3}$$

and showing that its limit as $h \to 0$ is the number $f(x)$ for each x in (a, b).

When we replace $F(x + h)$ and $F(x)$ by their defining integrals, the numerator in Equation (3) becomes

$$F(x + h) - F(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt.$$

The Additivity Rule for integrals (Table 5.3, Rule 5) simplifies the right side to

$$\int_x^{x+h} f(t)\, dt,$$

so that Equation (3) becomes

$$\begin{aligned} \frac{F(x + h) - F(x)}{h} &= \frac{1}{h}[F(x + h) - F(x)] \\ &= \frac{1}{h}\int_x^{x+h} f(t)\, dt. \end{aligned} \tag{4}$$

According to the Mean Value Theorem for Definite Integrals, the value of the last expression in Equation (4) is one of the values taken on by f in the interval between x and $x + h$. That is, for some number c in this interval,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c). \tag{5}$$

As $h \to 0$, $x + h$ approaches x, forcing c to approach x also (because c is trapped between x and $x + h$). Since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\lim_{h\to 0} f(c) = f(x). \tag{6}$$

Going back to the beginning, then, we have

$$\begin{aligned}
\frac{dF}{dx} &= \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} && \text{Definition of derivative}\\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt && \text{Eq. (4)}\\
&= \lim_{h\to 0} f(c) && \text{Eq. (5)}\\
&= f(x). && \text{Eq. (6)}
\end{aligned}$$

If $x = a$ or b, then the limit of Equation (3) is interpreted as a one-sided limit with $h \to 0^+$ or $h \to 0^-$, respectively. Then Theorem 1 in Section 3.1 shows that F is continuous for every point of $[a, b]$. This concludes the proof. ■

Fundamental Theorem, Part 2 (The Evaluation Theorem)

We now come to the second part of the Fundamental Theorem of Calculus. This part describes how to evaluate definite integrals without having to calculate limits of Riemann sums. Instead we find and evaluate an antiderivative at the upper and lower limits of integration.

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2
If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\,dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C for $a < x < b$ (by Corollary 2 of the Mean Value Theorem for Derivatives, Section 4.2). Since both F and G are continuous on $[a, b]$, we see that $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(t)\,dt. \end{aligned}$$ ■

The theorem says that to calculate the definite integral of f over $[a, b]$ all we need to do is:

1. Find an antiderivative F of f, and
2. Calculate the number $\int_a^b f(x)\,dx = F(b) - F(a)$.

The usual notation for $F(b) - F(a)$ is

$$F(x)\Big]_a^b \qquad \text{or} \qquad \Big[F(x)\Big]_a^b,$$

depending on whether F has one or more terms.

EXAMPLE 5 Evaluating Integrals

You Try It

(a) $$\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = 0$$

(b) $$\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^{0} = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

You Try It

(c) $$\begin{aligned} \int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\ &= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] \\ &= [8 + 1] - [5] = 4. \end{aligned}$$ ■

The process used in Example 5 was much easier than a Riemann sum computation.

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{dF}{dx} = f(x),$$

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, the equation

$$\int_a^x \frac{dF}{dt}\,dt = \int_a^x f(t)\,dt = F(x) - F(a)$$

says that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are "inverses" of each other. The Fundamental Theorem also says that every continuous function f has an antiderivative F. And it says that the differential equation $dy/dx = f(x)$ has a solution (namely, the function $y = F(x)$) for every continuous function f.

Total Area

The Riemann sum contains terms such as $f(c_k)\,\Delta_k$ which give the area of a rectangle when $f(c_k)$ is positive. When $f(c_k)$ is negative, then the product $f(c_k)\,\Delta_k$ is the negative of the rectangle's area. When we add up such terms for a negative function we get the negative of the area between the curve and the x-axis. If we then take the absolute value, we obtain the correct positive area.

EXAMPLE 6 Finding Area Using Antiderivatives

Calculate the area bounded by the x-axis and the parabola $y = 6 - x - x^2$.

Solution We find where the curve crosses the x-axis by setting

$$y = 0 = 6 - x - x^2 = (3 + x)(2 - x),$$

which gives

$$x = -3 \qquad \text{or} \qquad x = 2.$$

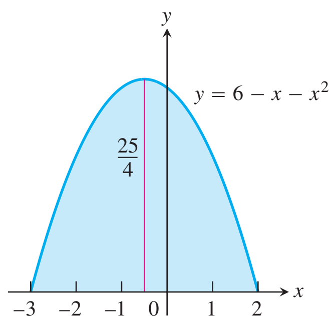

FIGURE 5.21 The area of this parabolic arch is calculated with a definite integral (Example 6).

The curve is sketched in Figure 5.21, and is nonnegative on $[-3, 2]$.

The area is

$$\int_{-3}^{2} (6 - x - x^2)\,dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$
$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right) = 20\tfrac{5}{6}.$$

The curve in Figure 5.21 is an arch of a parabola, and it is interesting to note that the area under such an arch is exactly equal to two-thirds the base times the altitude:

$$\frac{2}{3}(5)\left(\frac{25}{4}\right) = \frac{125}{6} = 20\tfrac{5}{6}.$$ ■

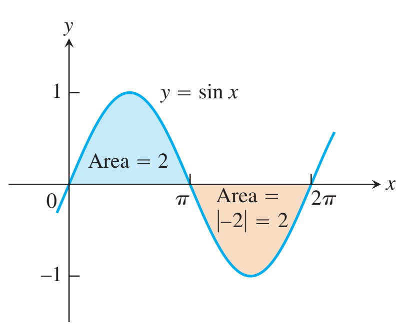

FIGURE 5.22 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

To compute the area of the region bounded by the graph of a function $y = f(x)$ and the x-axis requires more care when the function takes on both positive and negative values. We must be careful to break up the interval $[a, b]$ into subintervals on which the function doesn't change sign. Otherwise we might get cancellation between positive and negative signed areas, leading to an incorrect total. The correct total area is obtained by adding the absolute value of the definite integral over each subinterval where $f(x)$ does not change sign. The term "area" will be taken to mean *total area*.

EXAMPLE 7 Canceling Areas

Figure 5.22 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

Solution The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The definite integral is zero because the portions of the graph above and below the x-axis make canceling contributions.

The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2.$$

$$\int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2.$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$\text{Area} = |2| + |-2| = 4.$$

■

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

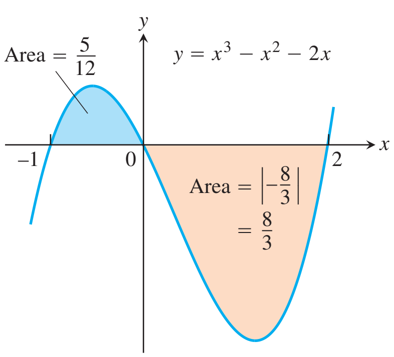

FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

EXAMPLE 8 Finding Area Using Antiderivatives

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0, -1$, and 2 (Figure 5.23). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$, and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals,

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

■