EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–26.

1. $\int_{-2}^{0} (2x + 5)\, dx$
2. $\int_{-3}^{4} \left(5 - \frac{x}{2}\right) dx$
3. $\int_{0}^{4} \left(3x - \frac{x^3}{4}\right) dx$
4. $\int_{-2}^{2} (x^3 - 2x + 3)\, dx$
5. $\int_{0}^{1} \left(x^2 + \sqrt{x}\right) dx$
6. $\int_{0}^{5} x^{3/2}\, dx$
7. $\int_{1}^{32} x^{-6/5}\, dx$
8. $\int_{-2}^{-1} \frac{2}{x^2}\, dx$
9. $\int_{0}^{\pi} \sin x\, dx$
10. $\int_{0}^{\pi} (1 + \cos x)\, dx$
11. $\int_{0}^{\pi/3} 2 \sec^2 x\, dx$
12. $\int_{\pi/6}^{5\pi/6} \csc^2 x\, dx$
13. $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta\, d\theta$
14. $\int_{0}^{\pi/3} 4 \sec u \tan u\, du$
15. $\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2}\, dt$
16. $\int_{-\pi/3}^{\pi/3} \frac{1 - \cos 2t}{2}\, dt$
17. $\int_{-\pi/2}^{\pi/2} (8y^2 + \sin y)\, dy$
18. $\int_{-\pi/3}^{-\pi/4} \left(4 \sec^2 t + \frac{\pi}{t^2}\right) dt$
19. $\int_{1}^{-1} (r + 1)^2\, dr$
20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t + 1)(t^2 + 4)\, dt$
21. $\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$
22. $\int_{1/2}^{1} \left(\frac{1}{v^3} - \frac{1}{v^4}\right) dv$
23. $\int_{1}^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\, ds$
24. $\int_{9}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}}\, du$
25. $\int_{-4}^{4} |x|\, dx$
26. $\int_{0}^{\pi} \frac{1}{2}(\cos x + |\cos x|)\, dx$

Derivatives of Integrals

Find the derivatives in Exercises 27–30

a. by evaluating the integral and differentiating the result.

b. by differentiating the integral directly.

27. $\frac{d}{dx} \int_{0}^{\sqrt{x}} \cos t\, dt$
28. $\frac{d}{dx} \int_{1}^{\sin x} 3t^2\, dt$
29. $\frac{d}{dt} \int_{0}^{t^4} \sqrt{u}\, du$
30. $\frac{d}{d\theta} \int_{0}^{\tan \theta} \sec^2 y\, dy$

Find dy/dx in Exercises 31–36.

31. $y = \int_{0}^{x} \sqrt{1 + t^2}\, dt$
32. $y = \int_{1}^{x} \frac{1}{t}\, dt, \quad x > 0$
33. $y = \int_{\sqrt{x}}^{0} \sin (t^2)\, dt$
34. $y = \int_{0}^{x^2} \cos \sqrt{t}\, dt$
35. $y = \int_{0}^{\sin x} \frac{dt}{\sqrt{1 - t^2}}, \quad |x| < \frac{\pi}{2}$
36. $y = \int_{\tan x}^{0} \frac{dt}{1 + t^2}$

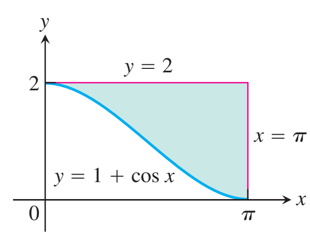
Area

In Exercises 37–42, find the total area between the region and the x-axis.

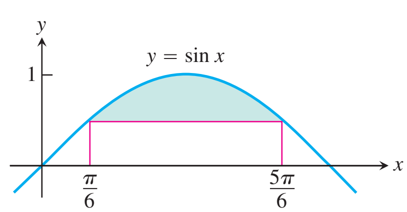
37. $y = -x^2 - 2x, \quad -3 \le x \le 2$
38. $y = 3x^2 - 3, \quad -2 \le x \le 2$
39. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$
40. $y = x^3 - 4x, \quad -2 \le x \le 2$
41. $y = x^{1/3}, \quad -1 \le x \le 8$
42. $y = x^{1/3} - x, \quad -1 \le x \le 8$

Find the areas of the shaded regions in Exercises 43–46.

43.



44.



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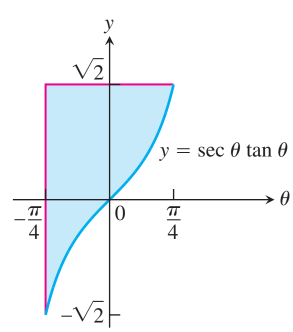
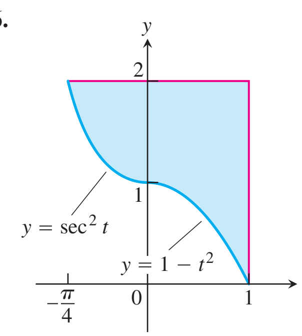


46.



Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 47–50. Which function solves which problem? Give brief reasons for your answers.

a. $y = \int_1^x \frac{1}{t}\,dt - 3$ **b.** $y = \int_0^x \sec t\,dt + 4$

c. $y = \int_{-1}^x \sec t\,dt + 4$ **d.** $y = \int_\pi^x \frac{1}{t}\,dt - 3$

47. $\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$ **48.** $y' = \sec x, \quad y(-1) = 4$

49. $y' = \sec x, \quad y(0) = 4$ **50.** $y' = \frac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 51–54 in terms of integrals.

51. $\frac{dy}{dx} = \sec x, \quad y(2) = 3$

52. $\frac{dy}{dx} = \sqrt{1 + x^2}, \quad y(1) = -2$

53. $\frac{ds}{dt} = f(t), \quad s(t_0) = s_0$

54. $\frac{dv}{dt} = g(t), \quad v(t_0) = v_0$

Applications

55. Archimedes' area formula for parabolas Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \le x \le b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

56. Revenue from marginal revenue Suppose that a company's marginal revenue from the manufacture and sale of egg beaters is

$$\frac{dr}{dx} = 2 - 2/(x + 1)^2,$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand egg beaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

57. Cost from marginal cost The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

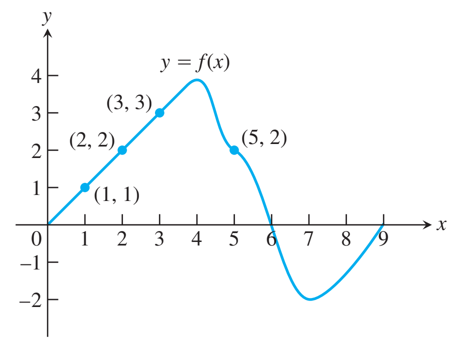
58. (*Continuation of Exercise 57.*) Find $c(400) - c(100)$, the cost of printing posters 101–400.

Drawing Conclusions About Motion from Graphs

59. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at time $t = 5$?

b. Is the acceleration of the particle at time $t = 5$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. At what time during the first 9 sec does s have its largest value?

e. Approximately when is the acceleration zero?

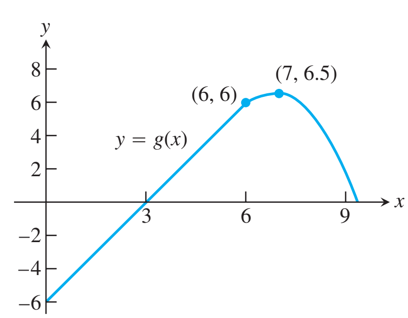
f. When is the particle moving toward the origin? away from the origin?

g. On which side of the origin does the particle lie at time $t = 9$?

60. Suppose that g is the differentiable function graphed here and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t g(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at $t = 3$?

b. Is the acceleration at time $t = 3$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. When does the particle pass through the origin?

e. When is the acceleration zero?

f. When is the particle moving away from the origin? toward the origin?

g. On which side of the origin does the particle lie at $t = 9$?

Theory and Examples

61. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

62. Find

$$\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t^2}{t^4+1}\,dt.$$

63. Suppose $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

64. Find $f(4)$ if $\int_0^x f(t)\,dt = x\cos\pi x$.

65. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\,dt$$

at $x = 1$.

66. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$$

at $x = -1$.

67. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a. g is a differentiable function of x.

b. g is a continuous function of x.

c. The graph of g has a horizontal tangent at $x = 1$.

d. g has a local maximum at $x = 1$.

e. g has a local minimum at $x = 1$.

f. The graph of g has an inflection point at $x = 1$.

g. The graph of dg/dx crosses the x-axis at $x = 1$.

68. Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a. h is a twice-differentiable function of x.

b. h and dh/dx are both continuous.

c. The graph of h has a horizontal tangent at $x = 1$.

d. h has a local maximum at $x = 1$.

e. h has a local minimum at $x = 1$.

f. The graph of h has an inflection point at $x = 1$.

g. The graph of dh/dx crosses the x-axis at $x = 1$.

T **69. The Fundamental Theorem** If f is continuous, we expect

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

to equal $f(x)$, as in the proof of Part 1 of the Fundamental Theorem. For instance, if $f(t) = \cos t$, then

$$\frac{1}{h}\int_x^{x+h} \cos t\,dt = \frac{\sin(x+h) - \sin x}{h}. \tag{7}$$

The right-hand side of Equation (7) is the difference quotient for the derivative of the sine, and we expect its limit as $h \to 0$ to be $\cos x$.

Graph $\cos x$ for $-\pi \le x \le 2\pi$. Then, in a different color if possible, graph the right-hand side of Equation (7) as a function of x for $h = 2, 1, 0.5$, and 0.1. Watch how the latter curves converge to the graph of the cosine as $h \to 0$.

T **70.** Repeat Exercise 69 for $f(t) = 3t^2$. What is

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} 3t^2\,dt = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}?$$

Graph $f(x) = 3x^2$ for $-1 \le x \le 1$. Then graph the quotient $((x+h)^3 - x^3)/h$ as a function of x for $h = 1, 0.5, 0.2$, and 0.1. Watch how the latter curves converge to the graph of $3x^2$ as $h \to 0$.

COMPUTER EXPLORATIONS

In Exercises 71–74, let $F(x) = \int_a^x f(t)\,dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

a. Plot the functions f and F together over $[a, b]$.

b. Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.

c. Over what intervals (approximately) is the function F increasing and decreasing? What is true about f over those intervals?

d. Calculate the derivative f' and plot it together with F. What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

71. $f(x) = x^3 - 4x^2 + 3x, \quad [0, 4]$

72. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12, \left[0, \frac{9}{2}\right]$

73. $f(x) = \sin 2x \cos \frac{x}{3}, \quad [0, 2\pi]$

74. $f(x) = x \cos \pi x, \quad [0, 2\pi]$

In Exercises 75–78, let $F(x) = \int_a^{u(x)} f(t)\, dt$ for the specified a, u, and f. Use a CAS to perform the following steps and answer the questions posed.

a. Find the domain of F.

b. Calculate $F'(x)$ and determine its zeros. For what points in its domain is F increasing? decreasing?

c. Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of F.

d. Using the information from parts (a)–(c), draw a rough hand-sketch of $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

75. $a = 1, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

76. $a = 0, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

77. $a = 0, \quad u(x) = 1 - x, \quad f(x) = x^2 - 2x - 3$

78. $a = 0, \quad u(x) = 1 - x^2, \quad f(x) = x^2 - 2x - 3$

In Exercises 79 and 80, assume that f is continuous and $u(x)$ is twice-differentiable.

79. Calculate $\frac{d}{dx}\int_a^{u(x)} f(t)\, dt$ and check your answer using a CAS.

80. Calculate $\frac{d^2}{dx^2}\int_a^{u(x)} f(t)\, dt$ and check your answer using a CAS.