

## 5.5

## Indefinite Integrals and the Substitution Rule

A definite integral is a number defined by taking the limit of Riemann sums associated with partitions of a finite closed interval whose norms go to zero. The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed easily if we can find an antiderivative of the function. Antiderivatives generally turn out to be more difficult to find than derivatives. However, it is well worth the effort to learn techniques for computing them.

Recall from Section 4.8 that the set of *all* antiderivatives of the function  $f$  is called the **indefinite integral** of  $f$  with respect to  $x$ , and is symbolized by

$$\int f(x) dx.$$

The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation. When finding the indefinite integral of a function  $f$ , remember that it always includes an arbitrary constant  $C$ .

We must distinguish carefully between definite and indefinite integrals. A definite integral  $\int_a^b f(x) dx$  is a *number*. An indefinite integral  $\int f(x) dx$  is a *function* plus an arbitrary constant  $C$ .

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives. The first integration techniques we develop are obtained by inverting rules for finding derivatives, such as the Power Rule and the Chain Rule.

### The Power Rule in Integral Form

If  $u$  is a differentiable function of  $x$  and  $n$  is a rational number different from  $-1$ , the Chain Rule tells us that

$$\frac{d}{dx} \left( \frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that  $u^{n+1}/(n+1)$  is one of the anti-derivatives of the function  $u^n(du/dx)$ . Therefore,

$$\int \left( u^n \frac{du}{dx} \right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler “differential” form,

$$\int u^n du,$$

obtained by treating the  $dx$ 's as differentials that cancel. We are thus led to the following rule.

If  $u$  is any differentiable function, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1, n \text{ rational}). \quad (1)$$

Equation (1) actually holds for any real exponent  $n \neq -1$ , as we see in Chapter 7.

In deriving Equation (1), we assumed  $u$  to be a differentiable function of the variable  $x$ , but the name of the variable does not matter and does not appear in the final formula. We could have represented the variable with  $\theta$ ,  $t$ ,  $y$ , or any other letter. Equation (1) says that whenever we can cast an integral in the form

$$\int u^n du, \quad (n \neq -1),$$

with  $u$  a differentiable function and  $du$  its differential, we can evaluate the integral as  $[u^{n+1}/(n+1)] + C$ .

### EXAMPLE 1 Using the Power Rule

$$\begin{aligned} \int \sqrt{1+y^2} \cdot 2y \, dy &= \int \sqrt{u} \cdot \left( \frac{du}{dy} \right) dy && \text{Let } u = 1 + y^2, \\ &= \int u^{1/2} du && du/dy = 2y \\ &= \frac{u^{(1/2)+1}}{(1/2)+1} + C && \text{Integrate, using Eq. (1)} \\ &= \frac{2}{3} u^{3/2} + C && \text{with } n = 1/2. \\ &= \frac{2}{3} (1+y^2)^{3/2} + C && \text{Simpler form} \\ & && \text{Replace } u \text{ by } 1 + y^2. \quad \blacksquare \end{aligned}$$

**EXAMPLE 2** Adjusting the Integrand by a Constant

$$\begin{aligned}
 \int \sqrt{4t-1} \, dt &= \int \frac{1}{4} \cdot \sqrt{4t-1} \cdot 4 \, dt \\
 &= \frac{1}{4} \int \sqrt{u} \cdot \left(\frac{du}{dt}\right) dt && \text{Let } u = 4t - 1, \\
 & && du/dt = 4. \\
 &= \frac{1}{4} \int u^{1/2} \, du && \text{With the } 1/4 \text{ out front,} \\
 & && \text{the integral is now in} \\
 & && \text{standard form.} \\
 &= \frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C && \text{Integrate, using Eq. (1)} \\
 & && \text{with } n = 1/2. \\
 &= \frac{1}{6} u^{3/2} + C && \text{Simpler form} \\
 &= \frac{1}{6} (4t-1)^{3/2} + C && \text{Replace } u \text{ by } 4t - 1. \quad \blacksquare
 \end{aligned}$$

**Substitution: Running the Chain Rule Backwards**

The substitutions in Examples 1 and 2 are instances of the following general rule.

**THEOREM 5** The Substitution Rule

If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du.$$

**Proof** The rule is true because, by the Chain Rule,  $F(g(x))$  is an antiderivative of  $f(g(x)) \cdot g'(x)$  whenever  $F$  is an antiderivative of  $f$ :

$$\begin{aligned}
 \frac{d}{dx} F(g(x)) &= F'(g(x)) \cdot g'(x) && \text{Chain Rule} \\
 &= f(g(x)) \cdot g'(x). && \text{Because } F' = f
 \end{aligned}$$

If we make the substitution  $u = g(x)$  then

$$\begin{aligned}
 \int f(g(x))g'(x) \, dx &= \int \frac{d}{dx} F(g(x)) \, dx \\
 &= F(g(x)) + C && \text{Fundamental Theorem} \\
 &= F(u) + C && u = g(x) \\
 &= \int F'(u) \, du && \text{Fundamental Theorem} \\
 &= \int f(u) \, du && F' = f \quad \blacksquare
 \end{aligned}$$

The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x))g'(x) dx,$$

when  $f$  and  $g'$  are continuous functions:

1. Substitute  $u = g(x)$  and  $du = g'(x) dx$  to obtain the integral

$$\int f(u) du.$$

2. Integrate with respect to  $u$ .
3. Replace  $u$  by  $g(x)$  in the result.

### EXAMPLE 3 Using Substitution

$$\begin{aligned} \int \cos(7\theta + 5) d\theta &= \int \cos u \cdot \frac{1}{7} du \\ &= \frac{1}{7} \int \cos u du \\ &= \frac{1}{7} \sin u + C \\ &= \frac{1}{7} \sin(7\theta + 5) + C \end{aligned}$$

Let  $u = 7\theta + 5$ ,  $du = 7 d\theta$ ,  
( $1/7$ )  $du = d\theta$ .

With the ( $1/7$ ) out front, the  
integral is now in standard form.

Integrate with respect to  $u$ ,  
Table 4.2.

Replace  $u$  by  $7\theta + 5$ .

We can verify this solution by differentiating and checking that we obtain the original function  $\cos(7\theta + 5)$ . ■

### EXAMPLE 4 Using Substitution

$$\begin{aligned} \int x^2 \sin(x^3) dx &= \int \sin(x^3) \cdot x^2 dx \\ &= \int \sin u \cdot \frac{1}{3} du \\ &= \frac{1}{3} \int \sin u du \\ &= \frac{1}{3} (-\cos u) + C \\ &= -\frac{1}{3} \cos(x^3) + C \end{aligned}$$

Let  $u = x^3$ ,  
 $du = 3x^2 dx$ ,  
( $1/3$ )  $du = x^2 dx$ .

Integrate with respect to  $u$ .

Replace  $u$  by  $x^3$ . ■

**EXAMPLE 5** Using Identities and Substitution

$$\begin{aligned}
 \int \frac{1}{\cos^2 2x} dx &= \int \sec^2 2x dx && \frac{1}{\cos 2x} = \sec 2x \\
 &= \int \sec^2 u \cdot \frac{1}{2} du && \begin{aligned} u &= 2x, \\ du &= 2 dx, \\ dx &= (1/2) du \end{aligned} \\
 &= \frac{1}{2} \int \sec^2 u du \\
 &= \frac{1}{2} \tan u + C && \frac{d}{du} \tan u = \sec^2 u \\
 &= \frac{1}{2} \tan 2x + C && u = 2x \quad \blacksquare
 \end{aligned}$$

The success of the substitution method depends on finding a substitution that changes an integral we cannot evaluate directly into one that we can. If the first substitution fails, try to simplify the integrand further with an additional substitution or two (see Exercises 49 and 50). Alternatively, we can start fresh. There can be more than one good way to start, as in the next example.

**EXAMPLE 6** Using Different Substitutions

Evaluate

$$\int \frac{2z dz}{\sqrt[3]{z^2 + 1}}.$$

**Solution** We can use the substitution method of integration as an exploratory tool: Substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try  $u = z^2 + 1$  or we might even press our luck and take  $u$  to be the entire cube root. Here is what happens in each case.

Solution 1: Substitute  $u = z^2 + 1$ .

$$\begin{aligned}
 \int \frac{2z dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{du}{u^{1/3}} && \begin{aligned} \text{Let } u &= z^2 + 1, \\ du &= 2z dz. \end{aligned} \\
 &= \int u^{-1/3} du && \text{In the form } \int u^n du \\
 &= \frac{u^{2/3}}{2/3} + C && \text{Integrate with respect to } u. \\
 &= \frac{3}{2} u^{2/3} + C \\
 &= \frac{3}{2} (z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } z^2 + 1.
 \end{aligned}$$

Solution 2: Substitute  $u = \sqrt[3]{z^2 + 1}$  instead.

$$\begin{aligned}\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{3u^2 \, du}{u} \\ &= 3 \int u \, du \\ &= 3 \cdot \frac{u^2}{2} + C \\ &= \frac{3}{2}(z^2 + 1)^{2/3} + C\end{aligned}$$

Let  $u = \sqrt[3]{z^2 + 1}$ ,  
 $u^3 = z^2 + 1$ ,  
 $3u^2 \, du = 2z \, dz$ .

Integrate with respect to  $u$ .

Replace  $u$  by  $(z^2 + 1)^{1/3}$ . ■

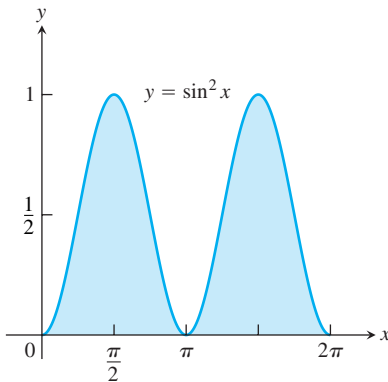
### The Integrals of $\sin^2 x$ and $\cos^2 x$

Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can use the substitution rule. Here is an example giving the integral formulas for  $\sin^2 x$  and  $\cos^2 x$  which arise frequently in applications.

#### EXAMPLE 7

$$\begin{aligned}\text{(a)} \quad \int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx && \sin^2 x = \frac{1 - \cos 2x}{2} \\ &= \frac{1}{2} \int (1 - \cos 2x) \, dx = \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx \\ &= \frac{1}{2}x - \frac{1}{2} \frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \int \cos^2 x \, dx &= \int \frac{1 + \cos 2x}{2} \, dx && \cos^2 x = \frac{1 + \cos 2x}{2} \\ &= \frac{x}{2} + \frac{\sin 2x}{4} + C && \text{As in part (a), but with a sign change}\end{aligned}$$



**FIGURE 5.24** The area beneath the curve  $y = \sin^2 x$  over  $[0, 2\pi]$  equals  $\pi$  square units (Example 8).

#### EXAMPLE 8 Area Beneath the Curve $y = \sin^2 x$

Figure 5.24 shows the graph of  $g(x) = \sin^2 x$  over the interval  $[0, 2\pi]$ . Find

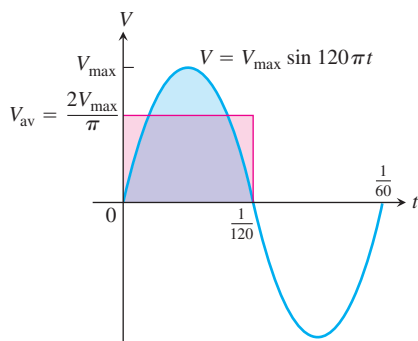
- the definite integral of  $g(x)$  over  $[0, 2\pi]$ .
- the area between the graph of the function and the  $x$ -axis over  $[0, 2\pi]$ .

#### Solution

(a) From Example 7(a), the definite integral is

$$\begin{aligned}\int_0^{2\pi} \sin^2 x \, dx &= \left[ \frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{2\pi} = \left[ \frac{2\pi}{2} - \frac{\sin 4\pi}{4} \right] - \left[ \frac{0}{2} - \frac{\sin 0}{4} \right] \\ &= [\pi - 0] - [0 - 0] = \pi.\end{aligned}$$

(b) The function  $\sin^2 x$  is nonnegative, so the area is equal to the definite integral, or  $\pi$ . ■



**FIGURE 5.25** The graph of the voltage  $V = V_{\max} \sin 120\pi t$  over a full cycle. Its average value over a half-cycle is  $2V_{\max}/\pi$ . Its average value over a full cycle is zero (Example 9).

### EXAMPLE 9 Household Electricity

We can model the voltage in our home wiring with the sine function

$$V = V_{\max} \sin 120\pi t,$$

which expresses the voltage  $V$  in volts as a function of time  $t$  in seconds. The function runs through 60 cycles each second (its frequency is 60 hertz, or 60 Hz). The positive constant  $V_{\max}$  (“vee max”) is the **peak voltage**.

The average value of  $V$  over the half-cycle from 0 to  $1/120$  sec (see Figure 5.25) is

$$\begin{aligned} V_{\text{av}} &= \frac{1}{(1/120) - 0} \int_0^{1/120} V_{\max} \sin 120\pi t \, dt \\ &= 120V_{\max} \left[ -\frac{1}{120\pi} \cos 120\pi t \right]_0^{1/120} \\ &= \frac{V_{\max}}{\pi} [-\cos \pi + \cos 0] \\ &= \frac{2V_{\max}}{\pi}. \end{aligned}$$

The average value of the voltage over a full cycle is zero, as we can see from Figure 5.25. (Also see Exercise 63.) If we measured the voltage with a standard moving-coil galvanometer, the meter would read zero.

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage, namely

$$V_{\text{rms}} = \sqrt{(V^2)_{\text{av}}}.$$

The subscript “rms” (read the letters separately) stands for “root mean square.” Since the average value of  $V^2 = (V_{\max})^2 \sin^2 120\pi t$  over a cycle is

$$(V^2)_{\text{av}} = \frac{1}{(1/60) - 0} \int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t \, dt = \frac{(V_{\max})^2}{2},$$

(Exercise 63, part c), the rms voltage is

$$V_{\text{rms}} = \sqrt{\frac{(V_{\max})^2}{2}} = \frac{V_{\max}}{\sqrt{2}}.$$

The values given for household currents and voltages are always rms values. Thus, “115 volts ac” means that the rms voltage is 115. The peak voltage, obtained from the last equation, is

$$V_{\max} = \sqrt{2} V_{\text{rms}} = \sqrt{2} \cdot 115 \approx 163 \text{ volts},$$

which is considerably higher. ■