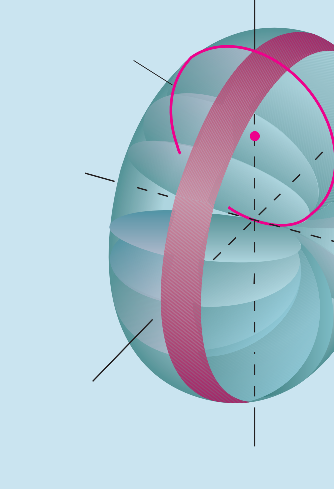


Chapter 6

APPLICATIONS OF DEFINITE INTEGRALS

OVERVIEW In Chapter 5 we discovered the connection between Riemann sums

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

associated with a partition P of the finite closed interval $[a, b]$ and the process of integration. We found that for a continuous function f on $[a, b]$, the limit of S_P as the norm of the partition $\|P\|$ approaches zero is the number

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f. We applied this to the problems of computing the area between the x-axis and the graph of $y = f(x)$ for $a \le x \le b$, and to finding the area between two curves.

In this chapter we extend the applications to finding volumes, lengths of plane curves, centers of mass, areas of surfaces of revolution, work, and fluid forces against planar walls. We define all these as limits of Riemann sums of continuous functions on closed intervals—that is, as definite integrals which can be evaluated using the Fundamental Theorem of Calculus.

6.1 Volumes by Slicing and Rotation About an Axis

In this section we define volumes of solids whose cross-sections are plane regions. A **cross-section** of a solid S is the plane region formed by intersecting S with a plane (Figure 6.1).

Suppose we want to find the volume of a solid S like the one in Figure 6.1. We begin by extending the definition of a cylinder from classical geometry to cylindrical solids with arbitrary bases (Figure 6.2). If the cylindrical solid has a known base area A and height h, then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

This equation forms the basis for defining the volumes of many solids that are not cylindrical by the *method of slicing*.

If the cross-section of the solid S at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$, and A is a continuous function of x, we can define and calculate the volume of the solid S as a definite integral in the following way.

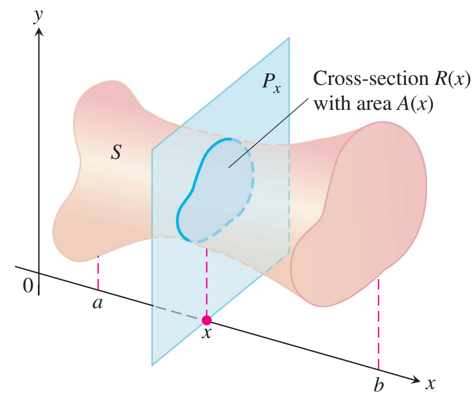


FIGURE 6.1 A cross-section of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

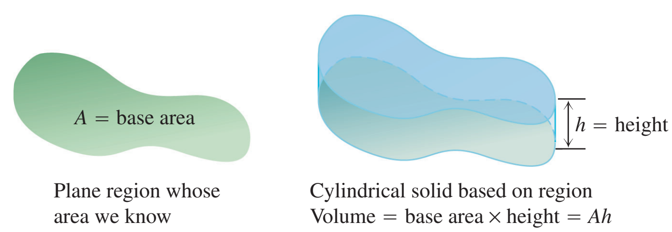


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$. The planes P_{x_k}, perpendicular to the x-axis at the partition points, slice S into thin "slabs" (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k)\,\Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero, so we define their limiting definite integral to be the volume of the solid S.

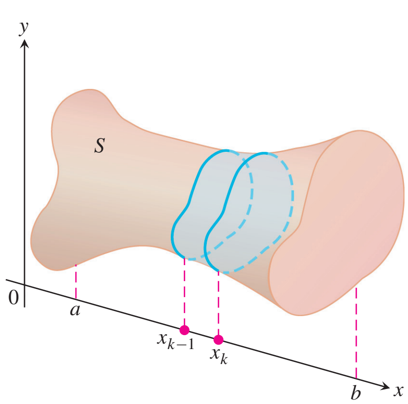


FIGURE 6.3 A typical thin slab in the solid S.

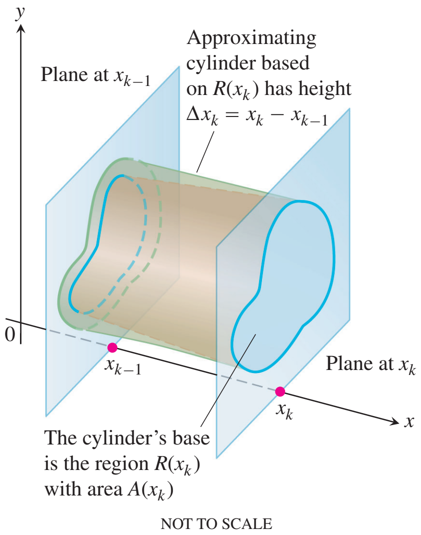


FIGURE 6.4 The solid thin slab in Figure 6.3 is approximated by the cylindrical solid with base $R(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

DEFINITION Volume

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\, dx.$$

This definition applies whenever $A(x)$ is continuous, or more generally, when it is integrable. To apply the formula in the definition to calculate the volume of a solid, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

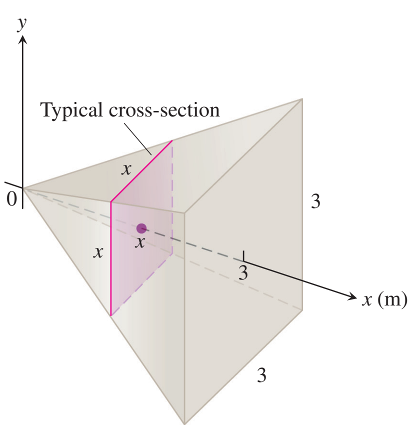


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

EXAMPLE 1 Volume of a Pyramid

You Try It

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

1. *A sketch.* We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross-section (Figure 6.5).
2. *A formula for* $A(x)$. The cross-section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$

3. *The limits of integration.* The squares lie on the planes from $x = 0$ to $x = 3$.
4. *Integrate to find the volume.*

$$V = \int_0^3 A(x)\, dx = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big]_0^3 = 9 \text{ m}^3$$ ■

HISTORICAL BIOGRAPHY

Bonaventura Cavalieri
(1598–1647)

Biography

EXAMPLE 2 Cavalieri's Principle

Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.6). This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids. ■

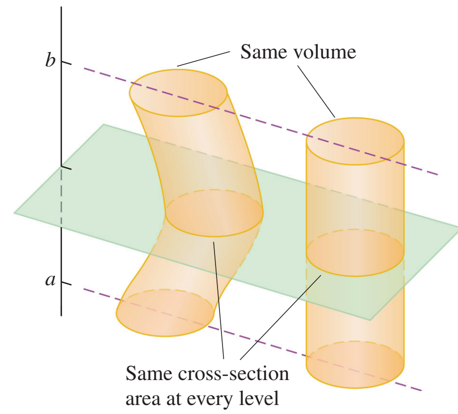


FIGURE 6.6 *Cavalieri's Principle:* These solids have the same volume, which can be illustrated with stacks of coins (Example 2).

EXAMPLE 3 Volume of a Wedge

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

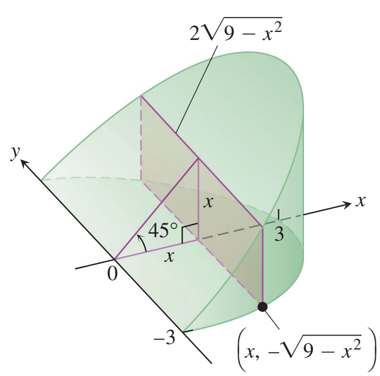


FIGURE 6.7 The wedge of Example 3, sliced perpendicular to the x-axis. The cross-sections are rectangles.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x-axis (Figure 6.7). The cross-section at x is a rectangle of area

$$\begin{aligned} A(x) &= (\text{height})(\text{width}) = (x)\left(2\sqrt{9-x^2}\right) \\ &= 2x\sqrt{9-x^2}\,. \end{aligned}$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$\begin{aligned} V &= \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx \\ &= -\frac{2}{3}(9-x^2)^{3/2}\Big]_0^3 \\ &= 0 + \frac{2}{3}(9)^{3/2} \\ &= 18. \end{aligned}$$

Let $u = 9 - x^2$, $du = -2x\,dx$, integrate, and substitute back.

■

Solids of Revolution: The Disk Method

The solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume gives

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

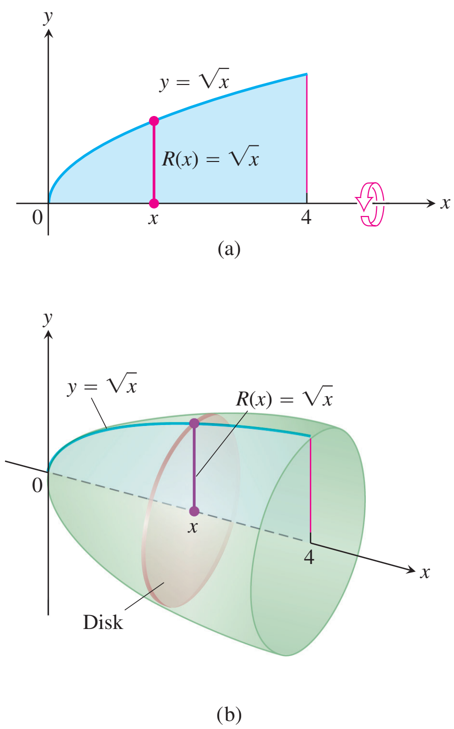


FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

EXAMPLE 4 A Solid of Revolution (Rotation About the x-Axis)

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2\, dx \\ &= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \qquad R(x) = \sqrt{x} \\ &= \pi \int_0^4 x\, dx = \pi \frac{x^2}{2}\bigg]_0^4 = \pi \frac{(4)^2}{2} = 8\pi. \end{aligned}$$

■

EXAMPLE 5 Volume of a Sphere

The circle

$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x-axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)\, dx = \int_{-a}^{a} \pi(a^2 - x^2)\, dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

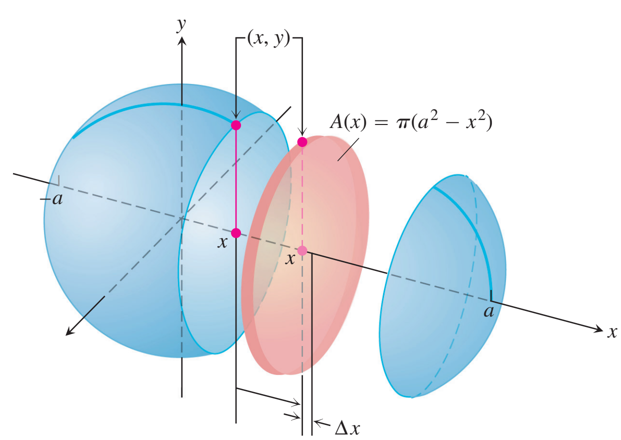


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

■

The axis of revolution in the next example is not the x-axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 6 A Solid of Revolution (Rotation About the Line $y = 1$)

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1, x = 4$ about the line $y = 1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(x)]^2\,dx \\ &= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx \\ &= \pi \int_1^4 \left[x - 2\sqrt{x} + 1\right] dx \\ &= \pi\left[\frac{x^2}{2} - 2 \cdot \frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}. \end{aligned}$$

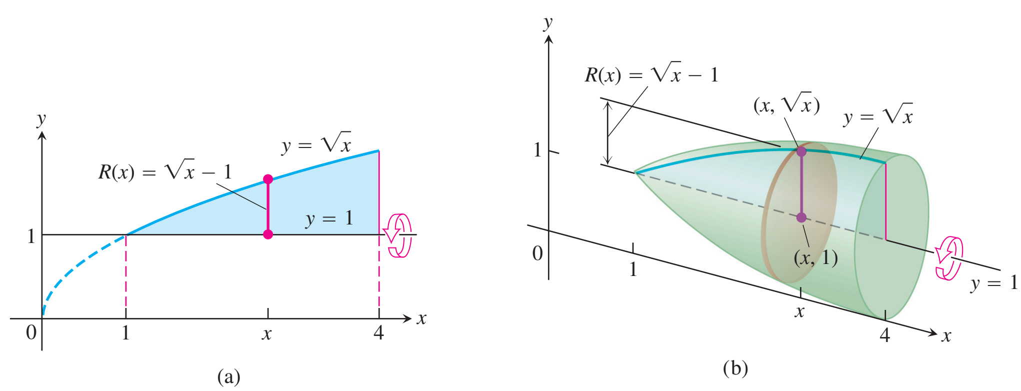


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6. ■

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \le y \le d$, about the y-axis, we use the same method with x replaced by y. In this case, the circular cross-section is

$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2.$$

EXAMPLE 7 Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

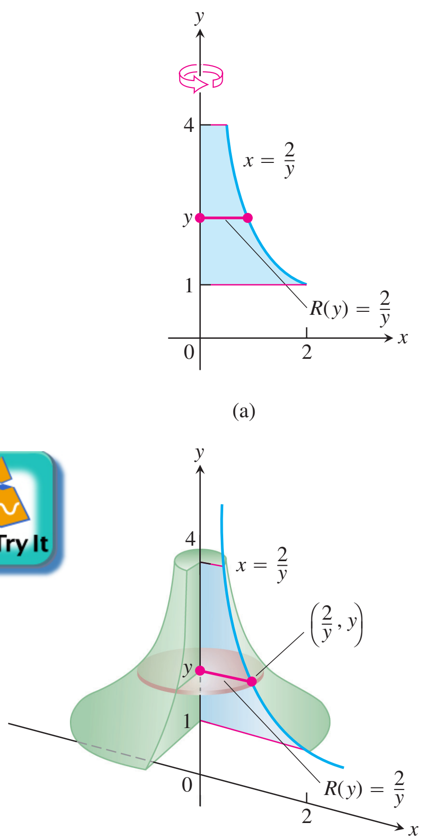


FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(y)]^2\, dy \\ &= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy \\ &= \pi\int_1^4 \frac{4}{y^2}\, dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right] \\ &= 3\pi. \end{aligned}$$

■

EXAMPLE 8 Rotation About a Vertical Axis

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$. The volume is

$$\begin{aligned} V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\, dy \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\, dy \qquad \begin{aligned} R(y) &= 3 - (y^2 + 1) \\ &= 2 - y^2 \end{aligned} \\ &= \pi\int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\, dy \\ &= \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \frac{64\pi\sqrt{2}}{15}. \end{aligned}$$

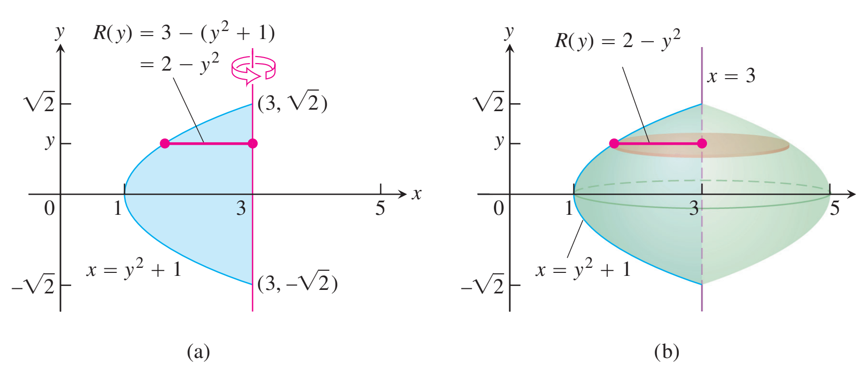


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8. ■

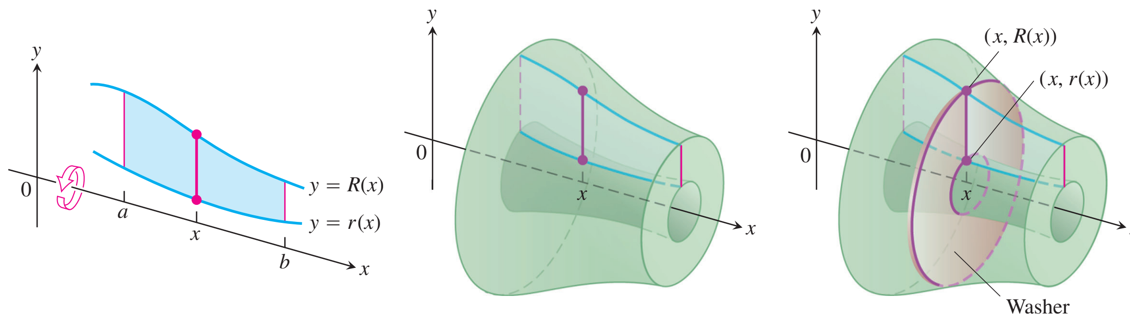


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

Outer radius:	$R(x)$
Inner radius:	$r(x)$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume gives

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

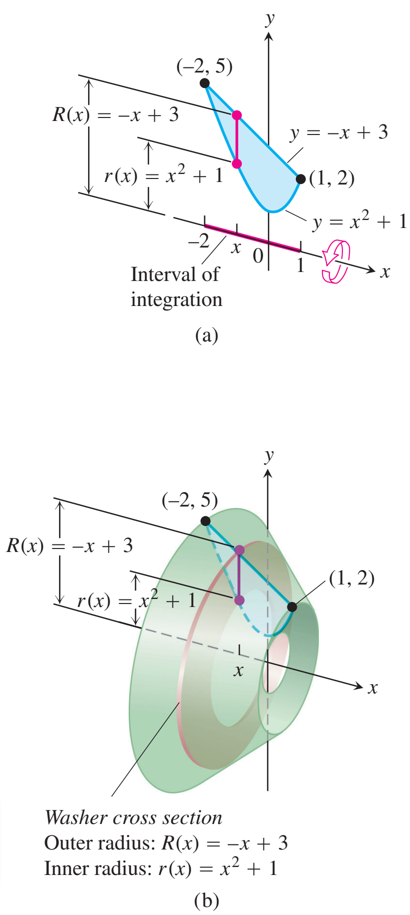


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

This method for calculating the volume of a solid of revolution is called the **washer method** because a slab is a circular washer of outer radius $R(x)$ and inner radius $r(x)$.

EXAMPLE 9 A Washer Cross-Section (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

These radii are the distances of the ends of the line segment from the axis of revolution (Figure 6.14).

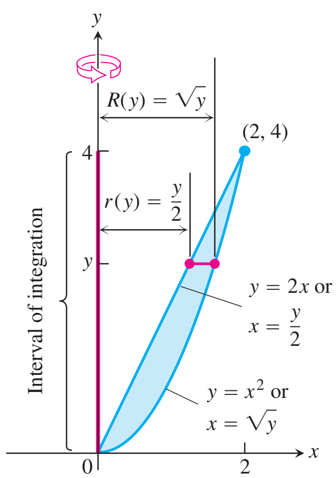
$$\text{Outer radius:}\quad R(x) = -x + 3$$
$$\text{Inner radius:}\quad r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ x = -2, &\quad x = 1 \end{aligned}$$

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx \\ &= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\,dx \qquad \text{Values from Steps 2 and 3} \\ &= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx \\ &= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \end{aligned}$$



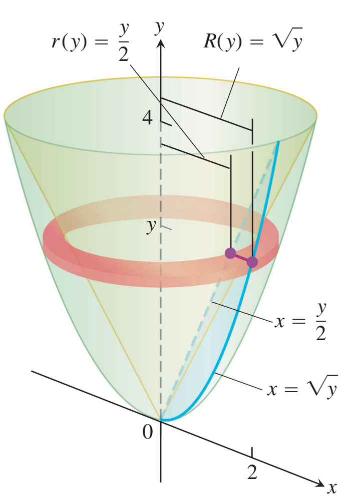


FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

To find the volume of a solid formed by revolving a region about the y-axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x. In this situation the line segment sweeping out a typical washer is perpendicular to the y-axis (the axis of revolution), and the outer and inner radii of the washer are functions of y.

EXAMPLE 10 A Washer Cross-Section (Rotation About the y-Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy \\ &= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy \\ &= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi. \end{aligned}$$

Summary

In all of our volume examples, no matter how the cross-sectional area $A(x)$ of a typical slab is determined, the definition of volume as the definite integral $V = \int_a^b A(x)\,dx$ is the heart of the calculations we made.