

6.3

Lengths of Plane Curves

HISTORICAL BIOGRAPHY

Archimedes
(287–212 B.C.)

We know what is meant by the length of a straight line segment, but without calculus, we have no precise notion of the length of a general winding curve. The idea of approximating the length of a curve running from point A to point B by subdividing the curve into many pieces and joining successive points of division by straight line segments dates back to the ancient Greeks. Archimedes used this method to approximate the circumference of a circle by inscribing a polygon of n sides and then using geometry to compute its perimeter

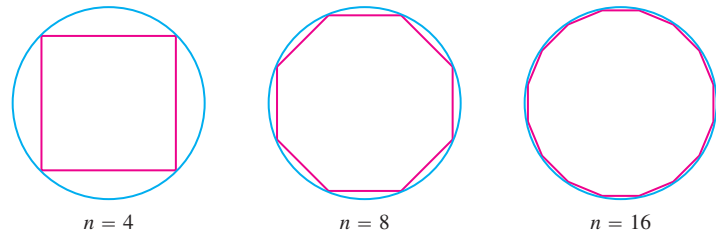


FIGURE 6.23 Archimedes used the perimeters of inscribed polygons to approximate the circumference of a circle. For $n = 96$ the approximation method gives $\pi \approx 3.14103$ as the circumference of the unit circle.

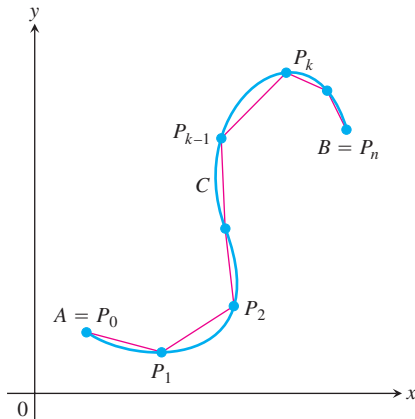


FIGURE 6.24 The curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1 , and so on, ending at $B = P_n$.

(Figure 6.23). The extension of this idea to a more general curve is displayed in Figure 6.24, and we now describe how that method works.

Length of a Parametrically Defined Curve

Let C be a curve given parametrically by the equations

$$x = f(t) \quad \text{and} \quad y = g(t), \quad a \leq t \leq b.$$

We assume the functions f and g have continuous derivatives on the interval $[a, b]$ that are not simultaneously zero. Such functions are said to be **continuously differentiable**, and the curve C defined by them is called a **smooth curve**. It may be helpful to imagine the curve as the path of a particle moving from point $A = (f(a), g(a))$ at time $t = a$ to point $B = (f(b), g(b))$ in Figure 6.24. We subdivide the path (or arc) AB into n pieces at points $A = P_0, P_1, P_2, \dots, P_n = B$. These points correspond to a partition of the interval $[a, b]$ by $a = t_0 < t_1 < t_2 < \dots < t_n = b$, where $P_k = (f(t_k), g(t_k))$. Join successive points of this subdivision by straight line segments (Figure 6.24). A representative line segment has length

$$\begin{aligned} L_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2} \end{aligned}$$

(see Figure 6.25). If Δt_k is small, the length L_k is approximately the length of arc $P_{k-1}P_k$. By the Mean Value Theorem there are numbers t_k^* and t_k^{**} in $[t_{k-1}, t_k]$ such that

$$\begin{aligned} \Delta x_k &= f(t_k) - f(t_{k-1}) = f'(t_k^*) \Delta t_k, \\ \Delta y_k &= g(t_k) - g(t_{k-1}) = g'(t_k^{**}) \Delta t_k. \end{aligned}$$

Assuming the path from A to B is traversed exactly once as t increases from $t = a$ to $t = b$, with no doubling back or retracing, an intuitive approximation to the “length” of the curve AB is the sum of all the lengths L_k :

$$\begin{aligned} \sum_{k=1}^n L_k &= \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sum_{k=1}^n \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2} \Delta t_k. \end{aligned}$$

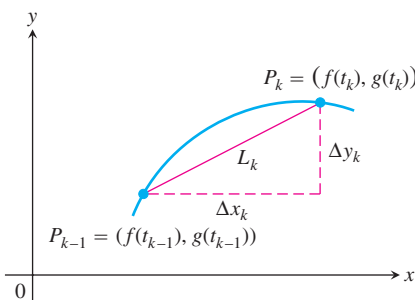


FIGURE 6.25 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

Although this last sum on the right is not exactly a Riemann sum (because f' and g' are evaluated at different points), a theorem in advanced calculus guarantees its limit, as the norm of the partition tends to zero, to be the definite integral

$$\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

Therefore, it is reasonable to define the length of the curve from A to B as this integral.

DEFINITION Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

A smooth curve C does not double back or reverse the direction of motion over the time interval $[a, b]$ since $(f')^2 + (g')^2 > 0$ throughout the interval.

If $x = f(t)$ and $y = g(t)$, then using the Leibniz notation we have the following result for arc length:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \quad (1)$$

What if there are two different parametrizations for a curve C whose length we want to find; does it matter which one we use? The answer, from advanced calculus, is no, as long as the parametrization we choose meets the conditions stated in the definition of the length of C (see Exercise 29).

EXAMPLE 1 The Circumference of a Circle

Find the length of the circle of radius r defined parametrically by

$$x = r \cos t \quad \text{and} \quad y = r \sin t, \quad 0 \leq t \leq 2\pi.$$

Solution As t varies from 0 to 2π , the circle is traversed exactly once, so the circumference is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

We find

$$\frac{dx}{dt} = -r \sin t, \quad \frac{dy}{dt} = r \cos t$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

So

$$L = \int_0^{2\pi} \sqrt{r^2} dt = r [t]_0^{2\pi} = 2\pi r. \quad \blacksquare$$

EXAMPLE 2 Applying the Parametric Formula for Length of a Curve

Find the length of the astroid (Figure 6.26)

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi.$$

Solution Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have

$$x = \cos^3 t, \quad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3 \cos^2 t(-\sin t)]^2 = 9 \cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3 \sin^2 t(\cos t)]^2 = 9 \sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9 \cos^2 t \sin^2 t (\underbrace{\cos^2 t + \sin^2 t}_1)}$$

$$= \sqrt{9 \cos^2 t \sin^2 t}$$

$$= 3 |\cos t \sin t|$$

$$= 3 \cos t \sin t.$$

$$\cos t \sin t \geq 0 \text{ for } 0 \leq t \leq \pi/2$$

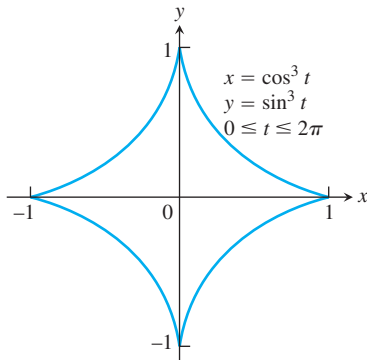


FIGURE 6.26 The astroid in Example 2.

Therefore,

$$\begin{aligned} \text{Length of first-quadrant portion} &= \int_0^{\pi/2} 3 \cos t \sin t dt \\ &= \frac{3}{2} \int_0^{\pi/2} \sin 2t dt && \cos t \sin t = (1/2) \sin 2t \\ &= -\frac{3}{4} \cos 2t \Big|_0^{\pi/2} = \frac{3}{2}. \end{aligned}$$

The length of the astroid is four times this: $4(3/2) = 6$. ■

HISTORICAL BIOGRAPHY

Gregory St. Vincent
(1584–1667)

Length of a Curve $y = f(x)$

Given a continuously differentiable function $y = f(x)$, $a \leq x \leq b$, we can assign $x = t$ as a parameter. The graph of the function f is then the curve C defined parametrically by

$$x = t \quad \text{and} \quad y = f(t), \quad a \leq t \leq b,$$

a special case of what we considered before. Then,

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = f'(t).$$

From our calculations in Section 3.5, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = f'(t)$$

giving

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 1 + [f'(t)]^2 \\ &= 1 + \left(\frac{dy}{dx}\right)^2 \\ &= 1 + [f'(x)]^2. \end{aligned}$$

Substitution into Equation (1) gives the arc length formula for the graph of $y = f(x)$.

Formula for the Length of $y = f(x)$, $a \leq x \leq b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + [f'(x)]^2} dx. \quad (2)$$

EXAMPLE 3 Applying the Arc Length Formula for a Graph

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1.$$

Solution We use Equation (2) with $a = 0$, $b = 1$, and

$$\begin{aligned} y &= \frac{4\sqrt{2}}{3}x^{3/2} - 1 \\ \frac{dy}{dx} &= \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2} \\ \left(\frac{dy}{dx}\right)^2 &= (2\sqrt{2}x^{1/2})^2 = 8x. \end{aligned}$$

The length of the curve from $x = 0$ to $x = 1$ is

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + 8x} dx \\ &= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big|_0^1 = \frac{13}{6}. \end{aligned}$$

Eq. (2) with
 $a = 0$, $b = 1$
Let $u = 1 + 8x$,
integrate, and
replace u by
 $1 + 8x$. ■

Dealing with Discontinuities in dy/dx

At a point on a curve where dy/dx fails to exist, dx/dy may exist and we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Equation (2):

Formula for the Length of $x = g(y)$, $c \leq y \leq d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d \sqrt{1 + [g'(y)]^2} dy. \quad (3)$$

EXAMPLE 4 Length of a Graph Which Has a Discontinuity in dy/dx

Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-1/3} \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (2).

We therefore rewrite the equation to express x in terms of y :

$$\begin{aligned} y &= \left(\frac{x}{2}\right)^{2/3} \\ y^{3/2} &= \frac{x}{2} && \text{Raise both sides} \\ &&& \text{to the power } 3/2. \\ x &= 2y^{3/2}. && \text{Solve for } x. \end{aligned}$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Figure 6.27).

The derivative

$$\frac{dx}{dy} = 2 \left(\frac{3}{2}\right) y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Equation (3) to find the curve's length:

$$\begin{aligned} L &= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + 9y} dy && \text{Eq. (3) with } \\ &= \frac{1}{9} \cdot \frac{2}{3} (1 + 9y)^{3/2} \Big|_0^1 && c = 0, d = 1. \\ &= \frac{2}{27} (10\sqrt{10} - 1) \approx 2.27. && \text{Let } u = 1 + 9y, \\ &&& du/9 = dy, \\ &&& \text{integrate, and} \\ &&& \text{substitute back.} \end{aligned}$$

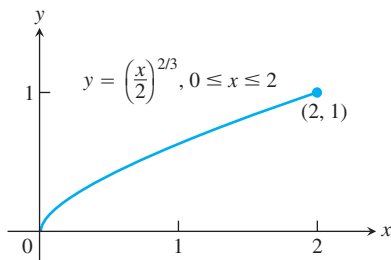


FIGURE 6.27 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

HISTORICAL BIOGRAPHY

James Gregory
(1638–1675)

The Short Differential Formula

Equation (1) is frequently written in terms of differentials in place of derivatives. This is done formally by writing $(dt)^2$ under the radical in place of the dt outside the radical, and then writing

$$\left(\frac{dx}{dt}\right)^2 (dt)^2 = \left(\frac{dx}{dt} dt\right)^2 = (dx)^2$$

and

$$\left(\frac{dy}{dt}\right)^2 (dt)^2 = \left(\frac{dy}{dt} dt\right)^2 = (dy)^2.$$

It is also customary to eliminate the parentheses in $(dx)^2$ and write dx^2 instead, so that Equation (1) is written

$$L = \int \sqrt{dx^2 + dy^2}. \quad (4)$$

We can think of these differentials as a way to summarize and simplify the properties of integrals. Differentials are given a precise mathematical definition in a more advanced text.

To do an integral computation, dx and dy must both be expressed in terms of one and the same variable, and appropriate limits must be supplied in Equation (4).

A useful way to remember Equation (4) is to write

$$ds = \sqrt{dx^2 + dy^2} \quad (5)$$

and treat ds as the differential of arc length, which can be integrated between appropriate limits to give the total length of a curve. Figure 6.28a gives the exact interpretation of ds corresponding to Equation (5). Figure 6.28b is not strictly accurate but is to be thought of as a simplified approximation of Figure 6.28a.

With Equation (5) in mind, the quickest way to recall the formulas for arc length is to remember the equation

$$\text{Arc length} = \int ds.$$

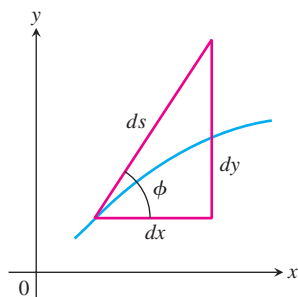
If we write $L = \int ds$ and have the graph of $y = f(x)$, we can rewrite Equation (5) to get

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \frac{dy^2}{dx^2} dx^2} = \sqrt{1 + \frac{dy^2}{dx^2}} dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

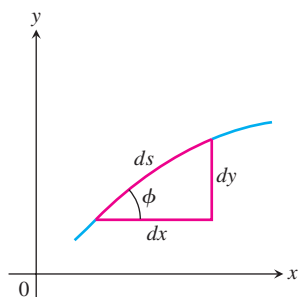
resulting in Equation (2). If we have instead $x = g(y)$, we rewrite Equation (5)

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dy^2 + \frac{dx^2}{dy^2} dy^2} = \sqrt{1 + \frac{dx^2}{dy^2}} dy = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

and obtain Equation (3).



(a)



(b)

FIGURE 6.28 Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.