6.5 Areas of Surfaces of Revolution and the Theorems of Pappus

When you jump rope, the rope sweeps out a surface in the space around you called a *surface of revolution*. The "area" of this surface depends on the length of the rope and the distance of each of its segments from the axis of revolution. In this section we define areas of surfaces of revolution. More complicated surfaces are treated in Chapter 16.

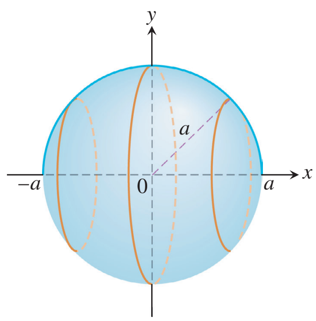


FIGURE 6.41 Rotating the semicircle $y = \sqrt{a^2 - x^2}$ of radius a with center at the origin generates a spherical surface with area $4\pi a^2$.

Defining Surface Area

We want our definition of the area of a surface of revolution to be consistent with known results from classical geometry for the surface areas of spheres, circular cylinders, and cones. So if the jump rope discussed in the introduction takes the shape of a semicircle with radius a rotated about the x-axis (Figure 6.41), it generates a sphere with surface area $4\pi a^2$.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the x-axis. If we rotate the horizontal line segment AB having length Δx about the x-axis (Figure 6.42a), we generate a cylinder with surface area $2\pi y \Delta x$. This area is the same as that of a rectangle with side lengths Δx and $2\pi y$ (Figure 6.42b). The length $2\pi y$ is the circumference of the circle of radius y generated by rotating the point (x, y) on the line AB about the x-axis.

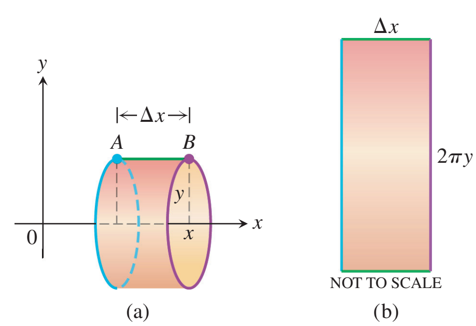


FIGURE 6.42 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x-axis has area $2\pi y \Delta x$. (b) The cut and rolled out cylindrical surface as a rectangle.

Suppose the line segment AB has length Δs and is slanted rather than horizontal. Now when AB is rotated about the x-axis, it generates a frustum of a cone (Figure 6.43a). From classical geometry, the surface area of this frustum is $2\pi y^* \Delta s$, where $y^* = (y_1 + y_2)/2$ is the average height of the slanted segment AB above the x-axis. This surface area is the same as that of a rectangle with side lengths Δs and $2\pi y^*$ (Figure 6.43b).

Let's build on these geometric principles to define the area of a surface swept out by revolving more general curves about the x-axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function $y = f(x)$, $a \le x \le b$, about the x-axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.44 shows a typical arc PQ and the band it sweeps out as part of the graph of f.

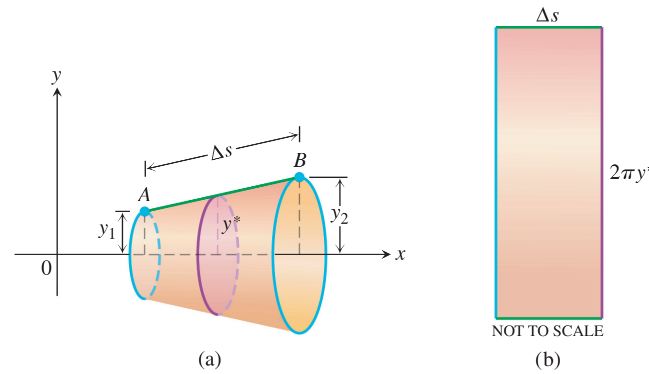


FIGURE 6.43 (a) The frustum of a cone generated by rotating the slanted line segment AB of length Δs about the x-axis has area $2\pi y^* \Delta s$. (b) The area of the rectangle for $y^* = \frac{y_1 + y_2}{2}$, the average height of AB above the x-axis.

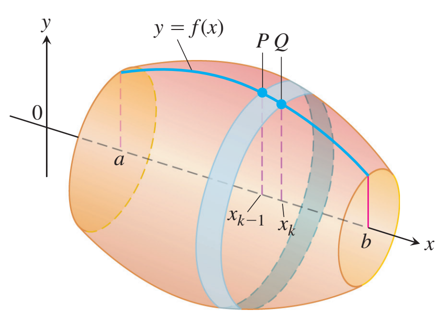


FIGURE 6.44 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \le x \le b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

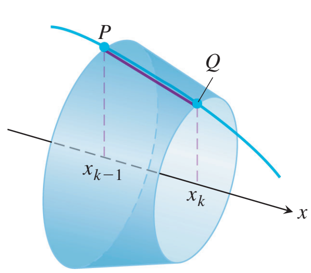


FIGURE 6.45 The line segment joining P and Q sweeps out a frustum of a cone.

As the arc PQ revolves about the x-axis, the line segment joining P and Q sweeps out a frustum of a cone whose axis lies along the x-axis (Figure 6.45). The surface area of this frustum approximates the surface area of the band swept out by the arc PQ. The surface area of the frustum of the cone shown in Figure 6.45 is $2\pi y^* L$, where y^* is the average height of the line segment joining P and Q, and L is its length (just as before). Since $f \ge 0$, from Figure 6.46 we see that the average height of the line segment is $y^* = (f(x_{k-1}) + f(x_k))/2$, and the slant length is $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. Therefore,

$$\begin{aligned}\text{Frustum surface area} &= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.\end{aligned}$$

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ, is approximated by the frustum area sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. Moreover, if the function f is differentiable, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Figure 6.47). At this point,

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k},$$

$$\Delta y_k = f'(c_k)\,\Delta x_k.$$

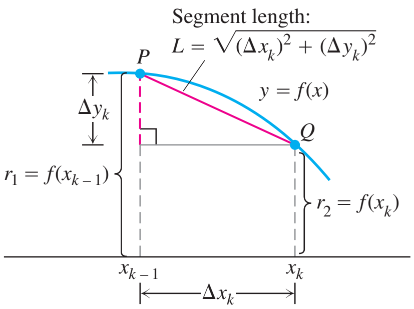


FIGURE 6.46 Dimensions associated with the arc and line segment PQ.

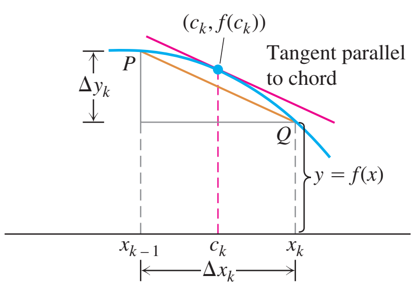


FIGURE 6.47 If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ.

With this substitution for Δy_k, the sums in Equation (1) take the form

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k)\,\Delta x_k)^2}$$

$$= \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2}\,\Delta x_k. \qquad (2)$$

These sums are not the Riemann sums of any function because the points x_{k-1}, x_k, and c_k are not the same. However, a theorem from advanced calculus assures us that as the norm of the partition of $[a, b]$ goes to zero, the sums in Equation (2) converge to the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of f from a to b.

DEFINITION Surface Area for Revolution About the *x*-Axis

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx. \qquad (3)$$

The square root in Equation (3) is the same one that appears in the formula for the length of the generating curve in Equation (2) of Section 6.3.

EXAMPLE 1 Applying the Surface Area Formula

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis (Figure 6.48).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.$$

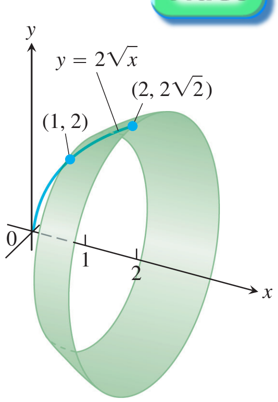


FIGURE 6.48 In Example 1 we calculate the area of this surface.

With these substitutions,

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}\,dx = 4\pi\int_1^2 \sqrt{x+1}\,dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$ ■

Revolution About the *y*-Axis

For revolution about the y-axis, we interchange x and y in Equation (3).

> **Surface Area for Revolution About the *y*-Axis**
> If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is
>
> $$S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy. \tag{4}$$

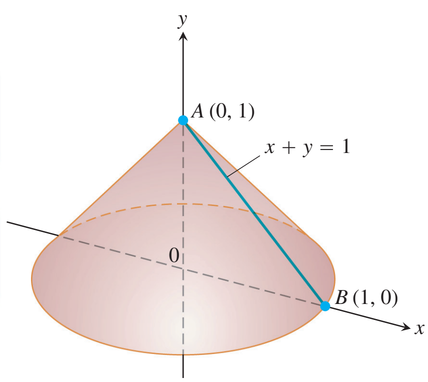


FIGURE 6.49 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

EXAMPLE 2 Finding Area for Revolution about the *y*-Axis

The line segment $x = 1 - y, 0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Figure 6.49. Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 2\pi(1 - y)\sqrt{2}\,dy$$

$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$

$$= \pi\sqrt{2}.$$

The results agree, as they should. ■

Parametrized Curves

Regardless of the coordinate axis of revolution, the square roots appearing in Equations (3) and (4) are the same ones that appear in the formulas for arc length in Section 6.3. If the curve is parametrized by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f and g are continuously differentiable on $[a, b]$, then the corresponding square root appearing in the arc length formula is

$$\sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

This observation leads to the following formulas for area of surfaces of revolution for smooth parametrized curves.

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis $(y \ge 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (5)$$

2. Revolution about the y-axis $(x \ge 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (6)$$

As with length, we can calculate surface area from any convenient parametrization that meets the stated criteria.

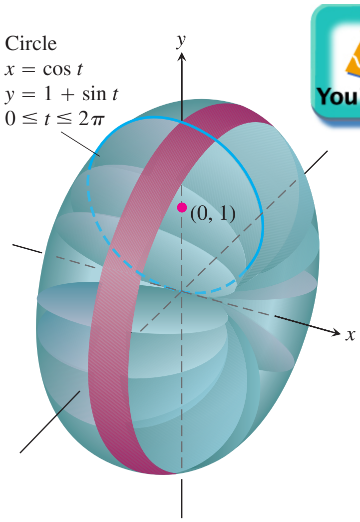


FIGURE 6.50 In Example 3 we calculate the area of the surface of revolution swept out by this parametrized curve.

You Try It

EXAMPLE 3 Applying Surface Area Formula

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis (Figure 6.50).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{Eq. (5) for revolution about the } x\text{-axis; } y = 1 + \sin t > 0$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$

$$= 2\pi \Big[t - \cos t\Big]_0^{2\pi} = 4\pi^2.$$

The Differential Form

The equations

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{and} \qquad S = \int_c^d 2\pi x \sqrt{\left(\frac{dx}{dy}\right)^2}\, dy$$

are often written in terms of the arc length differential $ds = \sqrt{dx^2 + dy^2}$ as

$$S = \int_a^b 2\pi y\, ds \qquad \text{and} \qquad S = \int_c^d 2\pi x\, ds.$$

In the first of these, y is the distance from the x-axis to an element of arc length ds. In the second, x is the distance from the y-axis to an element of arc length ds. Both integrals have the form

$$S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho\, ds \tag{7}$$

where ρ is the radius from the axis of revolution to an element of arc length ds (Figure 6.51).

In any particular problem, you would then express the radius function ρ and the arc length differential ds in terms of a common variable and supply limits of integration for that variable.

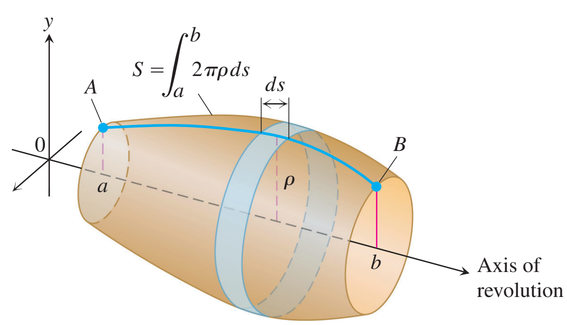


FIGURE 6.51 The area of the surface swept out by revolving arc AB about the axis shown here is $\int_a^b 2\pi\rho\, ds$. The exact expression depends on the formulas for ρ and ds.

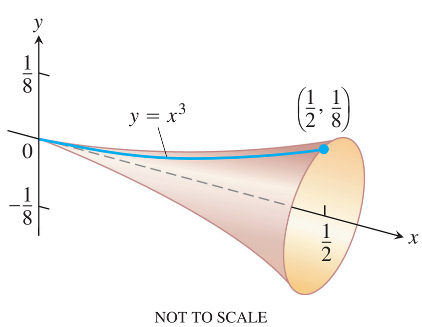


FIGURE 6.52 The surface generated by revolving the curve $y = x^3$, $0 \le x \le 1/2$, about the x-axis could be the design for a champagne glass (Example 4).

EXAMPLE 4 Using the Differential Form for Surface Areas

Find the area of the surface generated by revolving the curve $y = x^3$, $0 \le x \le 1/2$, about the x-axis (Figure 6.52).

Solution We start with the short differential form:

$$\begin{aligned} S &= \int 2\pi\rho\, ds \\ &= \int 2\pi y\, ds && \text{For revolution about the } x\text{-axis, the radius function is } \rho = y > 0 \text{ on } 0 \le x \le 1/2. \\ &= \int 2\pi y \sqrt{dx^2 + dy^2}. && ds = \sqrt{dx^2 + dy^2} \end{aligned}$$

We then decide whether to express dy in terms of dx or dx in terms of dy. The original form of the equation, $y = x^3$, makes it easier to express dy in terms of dx, so we continue the calculation with

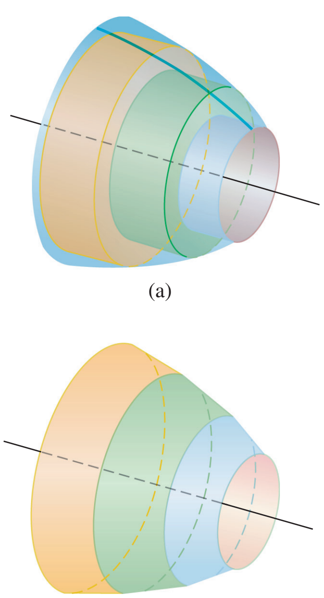
$$y = x^3, \qquad dy = 3x^2\,dx, \qquad \text{and} \qquad \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + (3x^2\,dx)^2}$$
$$= \sqrt{1 + 9x^4}\,dx.$$

With these substitutions, x becomes the variable of integration and

$$\begin{aligned} S &= \int_{x=0}^{x=1/2} 2\pi y \sqrt{dx^2 + dy^2} \\ &= \int_0^{1/2} 2\pi x^3 \sqrt{1 + 9x^4}\,dx \\ &= 2\pi\left(\frac{1}{36}\right)\left(\frac{2}{3}\right)(1 + 9x^4)^{3/2}\Big]_0^{1/2} \\ &= \frac{\pi}{27}\left[\left(1 + \frac{9}{16}\right)^{3/2} - 1\right] \\ &= \frac{\pi}{27}\left[\left(\frac{25}{16}\right)^{3/2} - 1\right] = \frac{\pi}{27}\left(\frac{125}{64} - 1\right) \\ &= \frac{61\pi}{1728}. \end{aligned}$$

Substitute $u = 1 + 9x^4$, $du/36 = x^3\,dx$; integrate, and substitute back.

Cylindrical Versus Conical Bands



(a)

(b)

FIGURE 6.53 Why not use (a) cylindrical bands instead of (b) conical bands to approximate surface area?

Why not find the surface area by approximating with cylindrical bands instead of conical bands, as suggested in Figure 6.53? The Riemann sums we get this way converge just as nicely as the ones based on conical bands, and the resulting integral is simpler. For revolution about the x-axis in this case, the radius in Equation (7) is $\rho = y$ and the band width is $ds = dx$. This leads to the integral formula

$$S = \int_a^b 2\pi f(x)\,dx \tag{8}$$

rather than the defining Equation (3). The problem with this new formula is that it fails to give results consistent with the surface area formulas from classical geometry, and that was one of our stated goals at the outset. Just because we end up with a nice-looking integral from a Riemann sum derivation does not mean it will calculate what we intend. (See Exercise 40.)

CAUTION Do not use Equation (8) to calculate surface area. It does *not* give the correct result.

The Theorems of Pappus

In the third century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids to surfaces and solids of revolution. The formulas provide shortcuts to a number of otherwise lengthy calculations.

THEOREM 1 Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \tag{9}$$

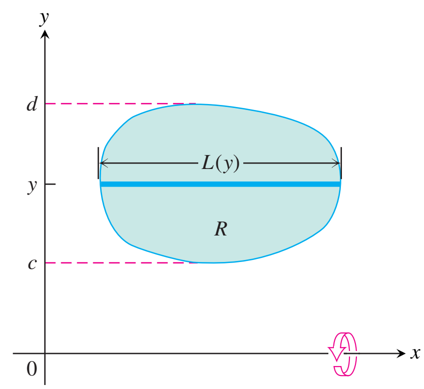


FIGURE 6.54 The region R is to be revolved (once) about the x-axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

Proof We draw the axis of revolution as the x-axis with the region R in the first quadrant (Figure 6.54). We let $L(y)$ denote the length of the cross-section of R perpendicular to the y-axis at y. We assume $L(y)$ to be continuous.

By the method of cylindrical shells, the volume of the solid generated by revolving the region about the x-axis is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\, dy = 2\pi \int_c^d y\, L(y)\, dy. \tag{10}$$

The y-coordinate of R's centroid is

$$\bar{y} = \frac{\int_c^d \tilde{y}\, dA}{A} = \frac{\int_c^d y\, L(y)\, dy}{A}, \qquad \tilde{y} = y,\ dA = L(y)dy$$

so that

$$\int_c^d y\, L(y)\, dy = A\bar{y}.$$

Substituting $A\bar{y}$ for the last integral in Equation (10) gives $V = 2\pi\bar{y}A$. With ρ equal to $\bar{y}$, we have $V = 2\pi\rho A$. ∎

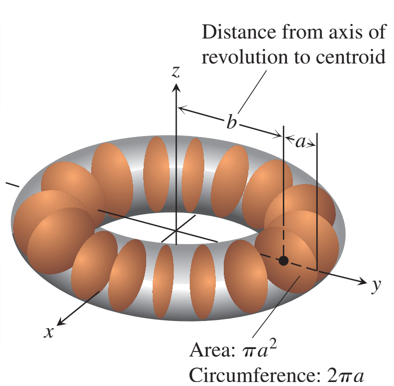


FIGURE 6.55 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 5).

EXAMPLE 5 Volume of a Torus

The volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center (Figure 6.55) is

$$V = 2\pi(b)(\pi a^2) = 2\pi^2 ba^2.$$ ■

EXAMPLE 6 Locate the Centroid of a Semicircular Region

Solution We model the region as the region between the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.56) and the x-axis and imagine revolving the region about the x-axis to generate a solid sphere. By symmetry, the x-coordinate of the centroid is $\bar{x} = 0$. With $\bar{y} = \rho$ in Equation (9), we have

$$\bar{y} = \frac{V}{2\pi A} = \frac{(4/3)\pi a^3}{2\pi(1/2)\pi a^2} = \frac{4}{3\pi} a.$$ ■

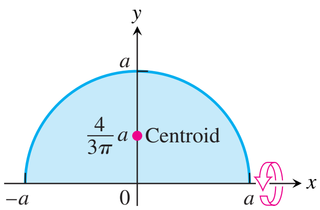


FIGURE 6.56 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 6).

THEOREM 2 Pappus's Theorem for Surface Areas
If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \tag{11}$$

The proof we give assumes that we can model the axis of revolution as the x-axis and the arc as the graph of a continuously differentiable function of x.

Proof We draw the axis of revolution as the x-axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Figure 6.57). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y\, ds = 2\pi \int_{x=a}^{x=b} y\, ds. \tag{12}$$

The y-coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \tilde{y}\, ds}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} y\, ds}{L}.$$

$L = \int ds$ is the arc's length and $\tilde{y} = y$.

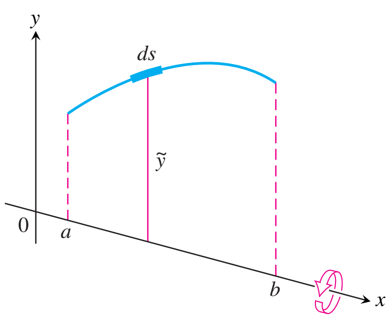


FIGURE 6.57 Figure for proving Pappus's area theorem.

Hence

$$\int_{x=a}^{x=b} y\, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Equation (12) gives $S = 2\pi\bar{y}L$. With ρ equal to $\bar{y}$, we have $S = 2\pi\rho L$. ■

EXAMPLE 7 Surface Area of a Torus

The surface area of the torus in Example 5 is

$$S = 2\pi(b)(2\pi a) = 4\pi^2 ba.$$ ■