

## 6.6

## Work

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on a body and the body's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing electrons together and lifting satellites into orbit.

### Work Done by a Constant Force

When a body moves a distance  $d$  along a straight line as a result of being acted on by a force of constant magnitude  $F$  in the direction of motion, we define the **work**  $W$  done by the force on the body with the formula

$$W = Fd \quad (\text{Constant-force formula for work}). \quad (1)$$

**Joules**

The joule, abbreviated J and pronounced “jewel,” is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In symbols,  $1 \text{ J} = 1 \text{ N} \cdot \text{m}$ .

From Equation (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International*, or International System), the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter ( $\text{N} \cdot \text{m}$ ). This combination appears so often it has a special name, the **joule**. In the British system, the unit of work is the foot-pound, a unit frequently used by engineers.

**EXAMPLE 1** Jacking Up a Car

If you jack up the side of a 2000-lb car 1.25 ft to change a tire (you have to apply a constant vertical force of about 1000 lb) you will perform  $1000 \times 1.25 = 1250$  ft-lb of work on the car. In SI units, you have applied a force of 4448 N through a distance of 0.381 m to do  $4448 \times 0.381 \approx 1695$  J of work. ■

**Work Done by a Variable Force Along a Line**

If the force you apply varies along the way, as it will if you are compressing a spring, the formula  $W = Fd$  has to be replaced by an integral formula that takes the variation in  $F$  into account.

Suppose that the force performing the work acts along a line that we take to be the  $x$ -axis and that its magnitude  $F$  is a continuous function of the position. We want to find the work done over the interval from  $x = a$  to  $x = b$ . We partition  $[a, b]$  in the usual way and choose an arbitrary point  $c_k$  in each subinterval  $[x_{k-1}, x_k]$ . If the subinterval is short enough,  $F$ , being continuous, will not vary much from  $x_{k-1}$  to  $x_k$ . The amount of work done across the interval will be about  $F(c_k)$  times the distance  $\Delta x_k$ , the same as it would be if  $F$  were constant and we could apply Equation (1). The total work done from  $a$  to  $b$  is therefore approximated by the Riemann sum

$$\text{Work} \approx \sum_{k=1}^n F(c_k) \Delta x_k.$$

We expect the approximation to improve as the norm of the partition goes to zero, so we define the work done by the force from  $a$  to  $b$  to be the integral of  $F$  from  $a$  to  $b$ .

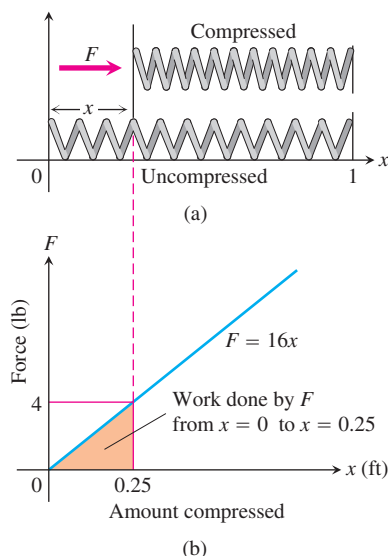
**DEFINITION** Work

The **work** done by a variable force  $F(x)$  directed along the  $x$ -axis from  $x = a$  to  $x = b$  is

$$W = \int_a^b F(x) dx. \quad (2)$$

The units of the integral are joules if  $F$  is in newtons and  $x$  is in meters, and foot-pounds if  $F$  is in pounds and  $x$  in feet. So, the work done by a force of  $F(x) = 1/x^2$  newtons along the  $x$ -axis from  $x = 1$  m to  $x = 10$  m is

$$W = \int_1^{10} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^{10} = -\frac{1}{10} + 1 = 0.9 \text{ J}.$$



**FIGURE 6.58** The force  $F$  needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

### Hooke's Law for Springs: $F = kx$

**Hooke's Law** says that the force it takes to stretch or compress a spring  $x$  length units from its natural (unstressed) length is proportional to  $x$ . In symbols,

$$F = kx. \quad (3)$$

The constant  $k$ , measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

#### EXAMPLE 2 Compressing a Spring

Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is  $k = 16$  lb/ft.

**Solution** We picture the uncompressed spring laid out along the  $x$ -axis with its movable end at the origin and its fixed end at  $x = 1$  ft (Figure 6.58). This enables us to describe the force required to compress the spring from 0 to  $x$  with the formula  $F = 16x$ . To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \quad \text{to} \quad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

The work done by  $F$  over this interval is

$$W = \int_0^{0.25} 16x \, dx = 8x^2 \Big|_0^{0.25} = 0.5 \text{ ft}\cdot\text{lb}. \quad \begin{array}{l} \text{Eq. (2) with} \\ a = 0, b = 0.25, \\ F(x) = 16x \end{array}$$

#### EXAMPLE 3 Stretching a Spring

A spring has a natural length of 1 m. A force of 24 N stretches the spring to a length of 1.8 m.

- Find the force constant  $k$ .
- How much work will it take to stretch the spring 2 m beyond its natural length?
- How far will a 45-N force stretch the spring?

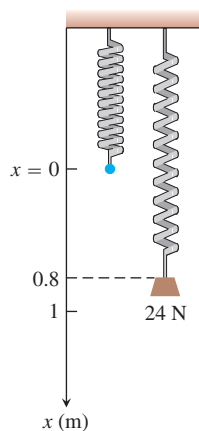
#### Solution

- (a) *The force constant.* We find the force constant from Equation (3). A force of 24 N stretches the spring 0.8 m, so

$$\begin{aligned} 24 &= k(0.8) && \text{Eq. (3) with} \\ k &= 24/0.8 = 30 \text{ N/m}. && F = 24, x = 0.8 \end{aligned}$$

- (b) *The work to stretch the spring 2 m.* We imagine the unstressed spring hanging along the  $x$ -axis with its free end at  $x = 0$  (Figure 6.59). The force required to stretch the spring  $x$  m beyond its natural length is the force required to pull the free end of the spring  $x$  units from the origin. Hooke's Law with  $k = 30$  says that this force is

$$F(x) = 30x.$$



**FIGURE 6.59** A 24-N weight stretches this spring 0.8 m beyond its unstressed length (Example 3).

The work done by  $F$  on the spring from  $x = 0$  m to  $x = 2$  m is

$$W = \int_0^2 30x \, dx = 15x^2 \Big|_0^2 = 60 \text{ J.}$$

(c) *How far will a 45-N force stretch the spring?* We substitute  $F = 45$  in the equation  $F = 30x$  to find

$$45 = 30x, \quad \text{or} \quad x = 1.5 \text{ m.}$$

A 45-N force will stretch the spring 1.5 m. No calculus is required to find this. ■

The work integral is useful to calculate the work done in lifting objects whose weights vary with their elevation.

#### EXAMPLE 4 Lifting a Rope and Bucket

A 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed (Figure 6.60). The rope weighs 0.08 lb/ft. How much work was spent lifting the bucket and rope?

**Solution** The bucket has constant weight so the work done lifting it alone is weight  $\times$  distance  $= 5 \cdot 20 = 100$  ft-lb.

The weight of the rope varies with the bucket's elevation, because less of it is freely hanging. When the bucket is  $x$  ft off the ground, the remaining proportion of the rope still being lifted weighs  $(0.08) \cdot (20 - x)$  lb. So the work in lifting the rope is

$$\begin{aligned} \text{Work on rope} &= \int_0^{20} (0.08)(20 - x) \, dx = \int_0^{20} (1.6 - 0.08x) \, dx \\ &= \left[ 1.6x - 0.04x^2 \right]_0^{20} = 32 - 16 = 16 \text{ ft-lb.} \end{aligned}$$

The total work for the bucket and rope combined is

$$100 + 16 = 116 \text{ ft-lb.} \quad \blacksquare$$

### Pumping Liquids from Containers

How much work does it take to pump all or part of the liquid from a container? To find out, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation  $W = Fd$  to each slab. We then evaluate the integral this leads to as the slabs become thinner and more numerous. The integral we get each time depends on the weight of the liquid and the dimensions of the container, but the way we find the integral is always the same. The next examples show what to do.

#### EXAMPLE 5 Pumping Oil from a Conical Tank

The conical tank in Figure 6.61 is filled to within 2 ft of the top with olive oil weighing  $57 \text{ lb/ft}^3$ . How much work does it take to pump the oil to the rim of the tank?

**Solution** We imagine the oil divided into thin slabs by planes perpendicular to the  $y$ -axis at the points of a partition of the interval  $[0, 8]$ .

The typical slab between the planes at  $y$  and  $y + \Delta y$  has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi \left( \frac{1}{2}y \right)^2 \Delta y = \frac{\pi}{4} y^2 \Delta y \text{ ft}^3.$$

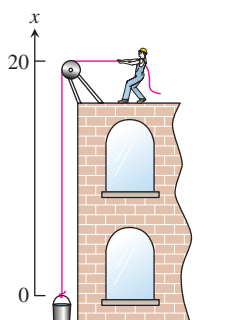


FIGURE 6.60 Lifting the bucket in Example 4.

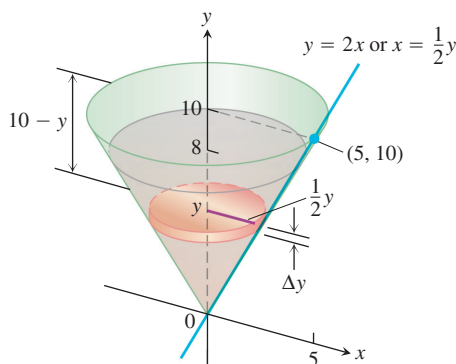


FIGURE 6.61 The olive oil and tank in Example 5.

The force  $F(y)$  required to lift this slab is equal to its weight,

$$F(y) = 57 \Delta V = \frac{57\pi}{4} y^2 \Delta y \text{ lb.} \quad \text{Weight} = \text{weight per unit volume} \times \text{volume}$$

The distance through which  $F(y)$  must act to lift this slab to the level of the rim of the cone is about  $(10 - y)$  ft, so the work done lifting the slab is about

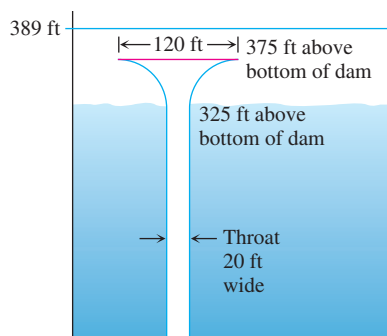
$$\Delta W = \frac{57\pi}{4} (10 - y)y^2 \Delta y \text{ ft-lb.}$$

Assuming there are  $n$  slabs associated with the partition of  $[0, 8]$ , and that  $y = y_k$  denotes the plane associated with the  $k$ th slab of thickness  $\Delta y_k$ , we can approximate the work done lifting all of the slabs with the Riemann sum

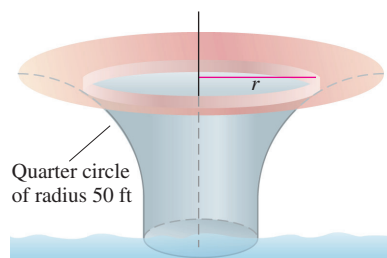
$$W \approx \sum_{k=1}^n \frac{57\pi}{4} (10 - y_k)y_k^2 \Delta y_k \text{ ft-lb.}$$

The work of pumping the oil to the rim is the limit of these sums as the norm of the partition goes to zero.

$$\begin{aligned} W &= \int_0^8 \frac{57\pi}{4} (10 - y)y^2 dy \\ &= \frac{57\pi}{4} \int_0^8 (10y^2 - y^3) dy \\ &= \frac{57\pi}{4} \left[ \frac{10y^3}{3} - \frac{y^4}{4} \right]_0^8 \approx 30,561 \text{ ft-lb.} \end{aligned}$$



(a)



(b)

**FIGURE 6.62** (a) Cross-section of the glory hole for a dam and (b) the top of the glory hole (Example 6).

### EXAMPLE 6 Pumping Water from a Glory Hole

A glory hole is a vertical drain pipe that keeps the water behind a dam from getting too high. The top of the glory hole for a dam is 14 ft below the top of the dam and 375 ft above the bottom (Figure 6.62). The hole needs to be pumped out from time to time to permit the removal of seasonal debris.

From the cross-section in Figure 6.62a, we see that the glory hole is a funnel-shaped drain. The throat of the funnel is 20 ft wide and the head is 120 ft across. The outside boundary of the head cross-section are quarter circles formed with 50-ft radii, shown in Figure 6.62b. The glory hole is formed by rotating a cross-section around its center. Consequently, all horizontal cross-sections are circular disks throughout the entire glory hole. We calculate the work required to pump water from

- the throat of the hole.
- the funnel portion.

#### Solution

- Pumping from the throat.* A typical slab in the throat between the planes at  $y$  and  $y + \Delta y$  has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi(10)^2 \Delta y \text{ ft}^3.$$

The force  $F(y)$  required to lift this slab is equal to its weight (about 62.4 lb/ft<sup>3</sup> for water),

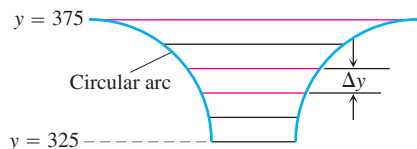
$$F(y) = 62.4 \Delta V = 6240\pi \Delta y \text{ lb.}$$

The distance through which  $F(y)$  must act to lift this slab to the top of the hole is  $(375 - y)$  ft, so the work done lifting the slab is

$$\Delta W = 6240\pi(375 - y) \Delta y \text{ ft-lb.}$$

We can approximate the work done in pumping the water from the throat by summing the work done lifting all the slabs individually, and then taking the limit of this Riemann sum as the norm of the partition goes to zero. This gives the integral

$$\begin{aligned} W &= \int_0^{325} 6240\pi(375 - y) dy \\ &= 6240\pi \left[ 375y - \frac{y^2}{2} \right]_0^{325} \\ &\approx 1,353,869,354 \text{ ft-lb.} \end{aligned}$$



**FIGURE 6.63** The glory hole funnel portion.

**(b) Pumping from the funnel.** To compute the work necessary to pump water from the funnel portion of the glory hole, from  $y = 325$  to  $y = 375$ , we need to compute  $\Delta V$  for approximating elements in the funnel as shown in Figure 6.63. As can be seen from the figure, the radii of the slabs vary with height  $y$ .

In Exercises 33 and 34, you are asked to complete the analysis to determine the total work required to pump the water and to find the power of the pumps necessary to pump out the glory hole. ■