

6.7 Fluid Pressures and Forces

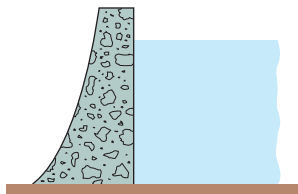


FIGURE 6.64 To withstand the increasing pressure, dams are built thicker as they go down.

We make dams thicker at the bottom than at the top (Figure 6.64) because the pressure against them increases with depth. The pressure at any point on a dam depends only on how far below the surface the point is and not on how much the surface of the dam happens to be tilted at that point. The pressure, in pounds per square foot at a point h feet below the surface, is always $62.4h$. The number 62.4 is the weight-density of water in pounds per cubic foot. The pressure h feet below the surface of any fluid is the fluid's *weight-density* times h .

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h :

$$p = wh. \quad (1)$$

In this section we use the equation $p = wh$ to derive a formula for the total force exerted by a fluid against all or part of a vertical or horizontal containing wall.

Weight-density

A fluid's weight-density is its weight per unit volume. Typical values (lb/ft³) are

Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Olive oil	57
Seawater	64
Water	62.4

The Constant-Depth Formula for Fluid Force

In a container of fluid with a flat horizontal base, the total force exerted by the fluid against the base can be calculated by multiplying the area of the base by the pressure at the base. We can do this because total force equals force per unit area (pressure) times area. (See Figure 6.65.) If F , p , and A are the total force, pressure, and area, then

$$\begin{aligned} F &= \text{total force} = \text{force per unit area} \times \text{area} \\ &= \text{pressure} \times \text{area} = pA \\ &= whA. \end{aligned} \quad \begin{array}{l} p = wh \text{ from} \\ \text{Eq. (1)} \end{array}$$

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \quad (2)$$

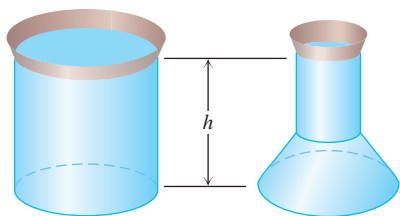


FIGURE 6.65 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

For example, the weight-density of water is 62.4 lb/ft³, so the fluid force at the bottom of a 10 ft \times 20 ft rectangular swimming pool 3 ft deep is

$$\begin{aligned} F &= whA = (62.4 \text{ lb/ft}^3)(3 \text{ ft})(10 \cdot 20 \text{ ft}^2) \\ &= 37,440 \text{ lb.} \end{aligned}$$

For a flat plate submerged *horizontally*, like the bottom of the swimming pool just discussed, the downward force acting on its upper face due to liquid pressure is given by Equation (2). If the plate is submerged *vertically*, however, then the pressure against it will be different at different depths and Equation (2) no longer is usable in that form (because h varies). By dividing the plate into many narrow horizontal bands or strips, we can create a Riemann sum whose limit is the fluid force against the side of the submerged vertical plate. Here is the procedure.

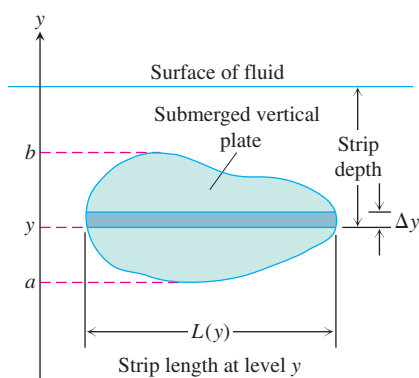


FIGURE 6.66 The force exerted by a fluid against one side of a thin, flat horizontal strip is about $\Delta F = \text{pressure} \times \text{area} = w \times (\text{strip depth}) \times L(y) \Delta y$.

The Variable-Depth Formula

Suppose we want to know the force exerted by a fluid against one side of a vertical plate submerged in a fluid of weight-density w . To find it, we model the plate as a region extending from $y = a$ to $y = b$ in the xy -plane (Figure 6.66). We partition $[a, b]$ in the usual way and imagine the region to be cut into thin horizontal strips by planes perpendicular to the y -axis at the partition points. The typical strip from y to $y + \Delta y$ is Δy units wide by $L(y)$ units long. We assume $L(y)$ to be a continuous function of y .

The pressure varies across the strip from top to bottom. If the strip is narrow enough, however, the pressure will remain close to its bottom-edge value of $w \times (\text{strip depth})$. The force exerted by the fluid against one side of the strip will be about

$$\Delta F = (\text{pressure along bottom edge}) \times (\text{area})$$

$$= w \cdot (\text{strip depth}) \cdot L(y) \Delta y.$$

Assume there are n strips associated with the partition of $a \leq y \leq b$ and that y_k is the bottom edge of the k th strip having length $L(y_k)$ and width Δy_k . The force against the entire plate is approximated by summing the forces against each strip, giving the Riemann sum

$$F \approx \sum_{k=1}^n (w \cdot (\text{strip depth})_k \cdot L(y_k)) \Delta y_k. \quad (3)$$

The sum in Equation (3) is a Riemann sum for a continuous function on $[a, b]$, and we expect the approximations to improve as the norm of the partition goes to zero. The force against the plate is the limit of these sums.

The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y -axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y . Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y) dy. \quad (4)$$

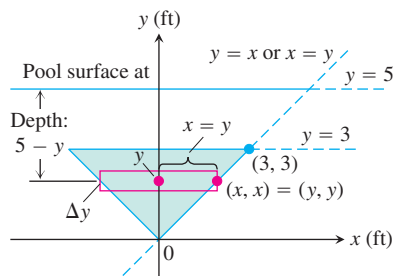


FIGURE 6.67 To find the force on one side of the submerged plate in Example 1, we can use a coordinate system like the one here.

EXAMPLE 1 Applying the Integral for Fluid Force

A flat isosceles right triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

Solution We establish a coordinate system to work in by placing the origin at the plate's bottom vertex and running the y -axis upward along the plate's axis of symmetry (Figure 6.67). The surface of the pool lies along the line $y = 5$ and the plate's top edge along the line $y = 3$. The plate's right-hand edge lies along the line $y = x$, with the upper right vertex at $(3, 3)$. The length of a thin strip at level y is

$$L(y) = 2x = 2y.$$

The depth of the strip beneath the surface is $(5 - y)$. The force exerted by the water against one side of the plate is therefore

$$\begin{aligned}
 F &= \int_a^b w \cdot \left(\begin{array}{c} \text{strip} \\ \text{depth} \end{array} \right) \cdot L(y) \, dy && \text{Eq. (4)} \\
 &= \int_0^3 62.4(5 - y)2y \, dy \\
 &= 124.8 \int_0^3 (5y - y^2) \, dy \\
 &= 124.8 \left[\frac{5}{2}y^2 - \frac{y^3}{3} \right]_0^3 = 1684.8 \text{ lb.}
 \end{aligned}$$

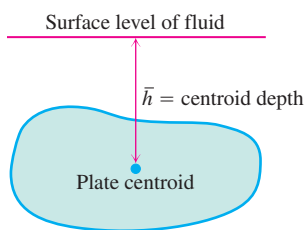


FIGURE 6.68 The force against one side of the plate is $w \cdot \bar{h} \cdot \text{plate area}$.

Fluid Forces and Centroids

If we know the location of the centroid of a submerged flat vertical plate (Figure 6.68), we can take a shortcut to find the force against one side of the plate. From Equation (4),

$$\begin{aligned}
 F &= \int_a^b w \times (\text{strip depth}) \times L(y) \, dy \\
 &= w \int_a^b (\text{strip depth}) \times L(y) \, dy \\
 &= w \times (\text{moment about surface level line of region occupied by plate}) \\
 &= w \times (\text{depth of plate's centroid}) \times (\text{area of plate}).
 \end{aligned}$$

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w , the distance \bar{h} from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A. \quad (5)$$

EXAMPLE 2 Finding Fluid Force Using Equation (5)

Use Equation (5) to find the force in Example 1.

Solution The centroid of the triangle (Figure 6.67) lies on the y -axis, one-third of the way from the base to the vertex, so $\bar{h} = 3$. The triangle's area is

$$\begin{aligned}
 A &= \frac{1}{2}(\text{base})(\text{height}) \\
 &= \frac{1}{2}(6)(3) = 9.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 F &= w\bar{h}A = (62.4)(3)(9) \\
 &= 1684.8 \text{ lb.}
 \end{aligned}$$