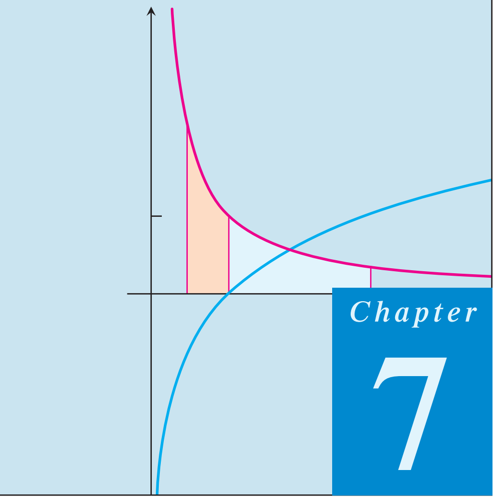


Chapter 7

TRANSCENDENTAL FUNCTIONS

OVERVIEW Functions can be classified into two broad groups (see Section 1.4). Polynomial functions are called *algebraic*, as are functions obtained from them by addition, multiplication, division, or taking powers and roots. Functions that are not algebraic are called *transcendental.* The trigonometric, exponential, logarithmic, and hyperbolic functions are transcendental, as are their inverses.

Transcendental functions occur frequently in many calculus settings and applications, including growths of populations, vibrations and waves, efficiencies of computer algorithms, and the stability of engineered structures. In this chapter we introduce several important transcendental functions and investigate their graphs, properties, derivatives, and integrals.

7.1 Inverse Functions and Their Derivatives

A function that undoes, or inverts, the effect of a function f is called the *inverse* of f. Many common functions, though not all, are paired with an inverse. Important inverse functions often show up in formulas for antiderivatives and solutions of differential equations. Inverse functions also play a key role in the development and properties of the logarithmic and exponential functions, as we will see in Section 7.3.

One-to-One Functions

A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one. These functions take on any one value in their range exactly once.

DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

EXAMPLE 1 Domains of One-to-One Functions

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. The sine *is* one-to-one on $[0, \pi/2]$, however, because it is a strictly increasing function on $[0, \pi/2]$. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If it intersects the line more than once, it assumes the same y-value more than once, and is therefore not one-to-one (Figure 7.1).

The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

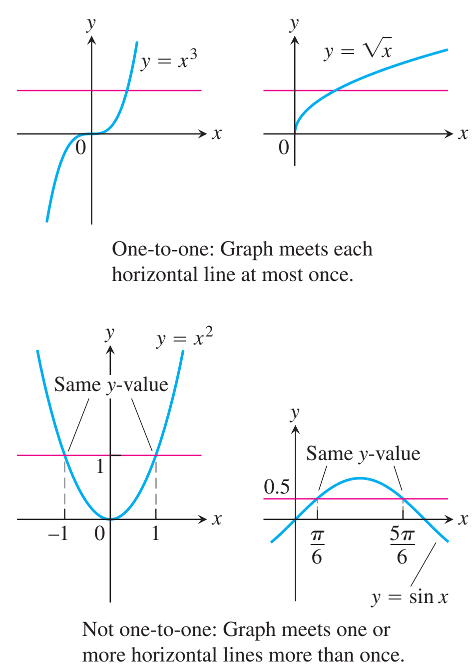


FIGURE 7.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$, but $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION **Inverse Function**

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The domains and ranges of f and f^{-1} are interchanged. The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent: $f^{-1}(x)$ does not mean $1/f(x)$.

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$ we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y with which we began. Composing a function and its inverse has the same effect as doing nothing.

$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$

$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f)$$

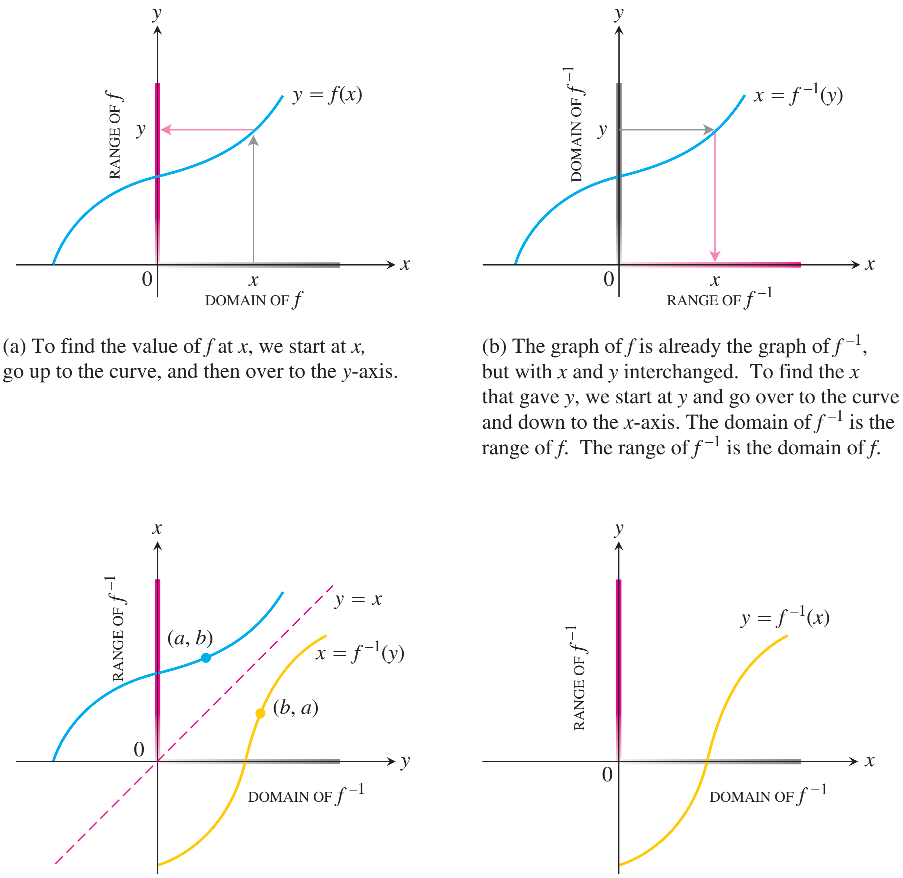
Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

A function that is increasing on an interval, satisfying $f(x_2) > f(x_1)$ when $x_2 > x_1$, is one-to-one and has an inverse. Decreasing functions also have an inverse (Exercise 39). Functions that have positive derivatives at all x are increasing (Corollary 3 of the Mean Value Theorem, Section 4.2), and so they have inverses. Similarly, functions with negative derivatives at all x are decreasing and have inverses. Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse, as with the function $\sec^{-1} x$ in Section 7.7.

Finding Inverses

The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing this process. Start with a point y on the y-axis, go horizontally to the graph, and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$ (Figure 7.2).

We want to set up the graph of f^{-1} so that its input values lie along the x-axis, as is usually done for functions, rather then on the y-axis. To achieve this we interchange the x and y axes by reflecting across the 45° line $y = x$. After this reflection we have a new graph that represents f^{-1}. The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x-axis, going vertically to the graph and then horizontally to the y-axis to get the value of $f^{-1}(x)$. Figure 7.2 indicates the relation between the graphs of f and f^{-1}. The graphs are interchanged by reflection through the line $y = x$.



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f is already the graph of f^{-1}, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the more usual way, we reflect the system in the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.

The process of passing from f to f^{-1} can be summarized as a two-step process.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE 2 Finding an Inverse Function

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:*

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

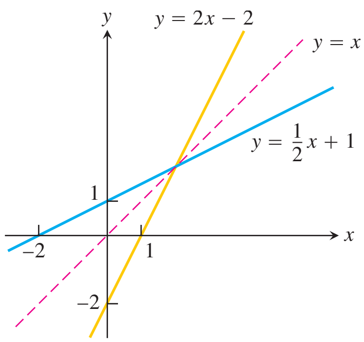


FIGURE 7.3 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other (Example 2).

2. *Interchange x and y:* $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

See Figure 7.3. ■

EXAMPLE 3 Finding an Inverse Function

Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

You Try It

Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 7.4).

Notice that, unlike the restricted function $y = x^2, x \geq 0$, the unrestricted function $y = x^2$ is not one-to-one and therefore has no inverse. ■

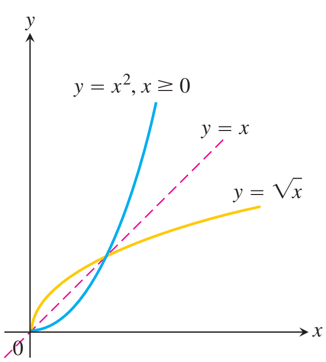


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 3).

Derivatives of Inverses of Differentiable Functions

If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$ from Example 2, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

The derivatives are reciprocals of one another. The graph of f is the line $y = (1/2)x + 1$, and the graph of f^{-1} is the line $y = 2x - 2$ (Figure 7.3). Their slopes are reciprocals of one another.

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$ (Figure 7.5), the reflected line has slope $1/m$ (Exercise 36).

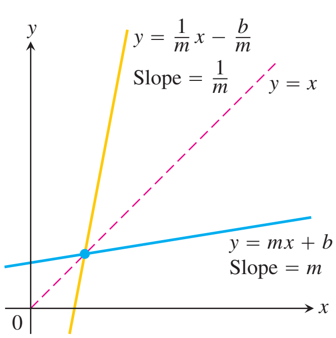


FIGURE 7.5 The slopes of nonvertical lines reflected through the line $y = x$ are reciprocals of each other.

The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$ (Figure 7.6). If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$ then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, and this infinite slope implies that f^{-1} is not differentiable

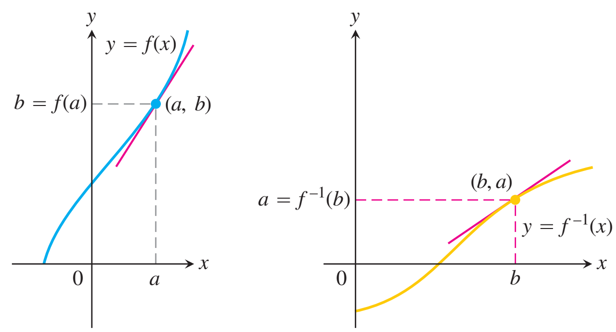


FIGURE 7.6 The graphs of inverse functions have reciprocal slopes at corresponding points.

at $f(a)$. Theorem 1 gives the conditions under which f^{-1} is differentiable in its domain, which is the same as the range of f.

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}} \tag{1}$$

The proof of Theorem 1 is omitted, but here is another way to view it. When $y = f(x)$ is differentiable at $x = a$ and we change x by a small amount dx, the corresponding change in y is approximately

$$dy = f'(a)\,dx.$$

This means that y changes about $f'(a)$ times as fast as x when $x = a$ and that x changes about $1/f'(a)$ times as fast as y when $y = b$. It is reasonable that the derivative of f^{-1} at b is the reciprocal of the derivative of f at a.

EXAMPLE 4 Applying Theorem 1

The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

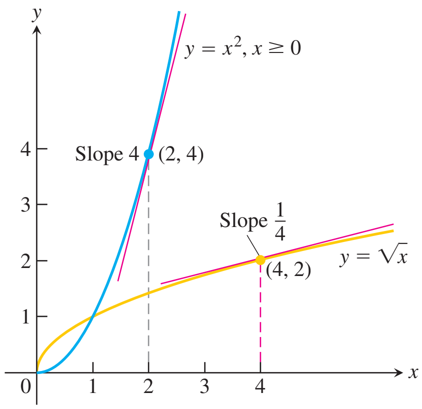


FIGURE 7.7 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 4).

Theorem 1 predicts that the derivative of $f^{-1}(x)$ is

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{2(f^{-1}(x))}\\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

Theorem 1 gives a derivative that agrees with our calculation using the Power Rule for the derivative of the square root function.

Let's examine Theorem 1 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 1 says that the derivative of f at 2, $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x}\bigg|_{x=2} = \frac{1}{4}.$$

See Figure 7.7. ■

Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

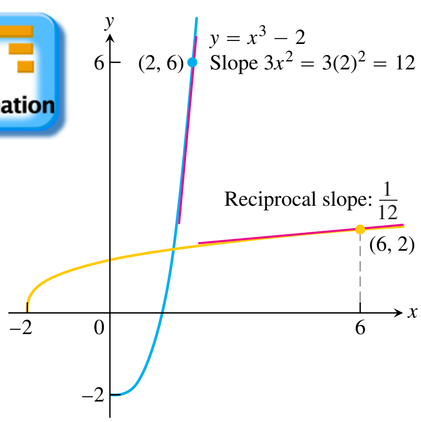


FIGURE 7.8 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 5).

EXAMPLE 5 Finding a Value of the Inverse Derivative

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

$$\frac{df}{dx}\bigg|_{x=2} = 3x^2\bigg|_{x=2} = 12$$

$$\frac{df^{-1}}{dx}\bigg|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12} \qquad \text{Eq. (1)}$$

See Figure 7.8. ■

Parametrizing Inverse Functions

We can graph or represent any function $y = f(x)$ parametrically as

$$x = t \quad \text{and} \quad y = f(t).$$

Interchanging t and $f(t)$ produces parametric equations for the inverse:

$$x = f(t) \quad \text{and} \quad y = t$$

(see Section 3.5).

For example, to graph the one-to-one function $f(x) = x^2, x \geq 0$, on a grapher together with its inverse and the line $y = x, x \geq 0$, use the parametric graphing option with

$$\begin{aligned}&\text{Graph of } f: && x_1 = t, && y_1 = t^2, && t \geq 0\\ &\text{Graph of } f^{-1}: && x_2 = t^2, && y_2 = t\\ &\text{Graph of } y = x: && x_3 = t, && y_3 = t\end{aligned}$$